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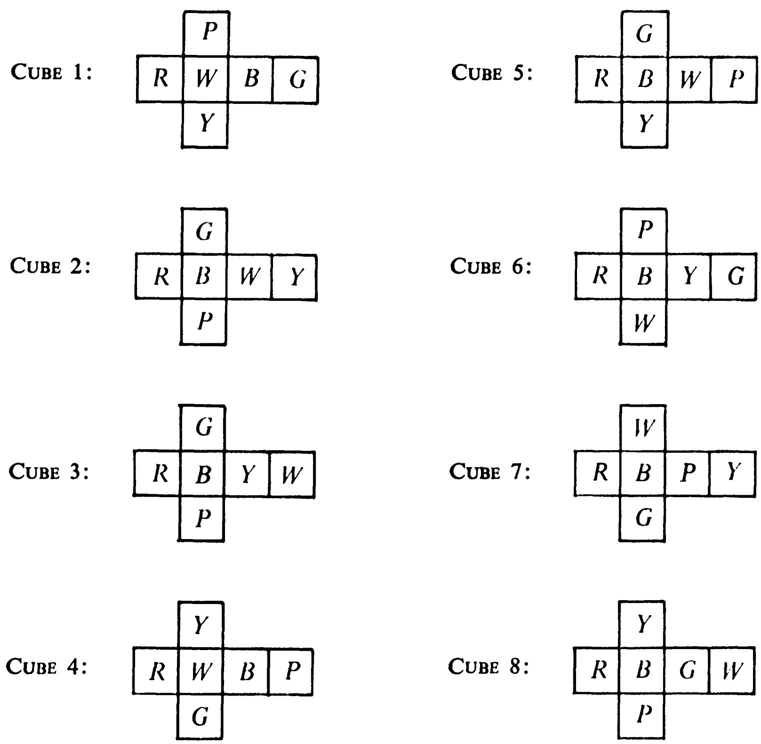
"EIGHT BLOCKS TO MADNESS" — A LOGICAL SOLUTION

STEVEN J. KAHAN, Laurelton, New York

"Arrange the eight small cubes into one large cube so that each of the six outside surfaces of the large cube has only one color on each side." This is the object of a challenging new puzzle, entitled "Eight Blocks to Madness", which has recently appeared on the market. As its title suggests, the puzzle consists of eight cubes, each having six different-colored faces. The object of this brief discussion is to present a

DIAGRAM 1.



KEY: *R* = red *W* = white *B* = blue *G* = green *P* = purple *Y* = yellow

simple and logical method for satisfying the required condition of the puzzle.

Diagram 1 gives the color configuration of the eight small cubes, where each cube is formed by folding along the interior lines.

When arranged into the large cube, each of the small cubes has exactly three visible faces. Referring to Diagram 2 we obtain Table 1, which associates each position in the large cube with the visible faces in that position:

DIAGRAM 2.

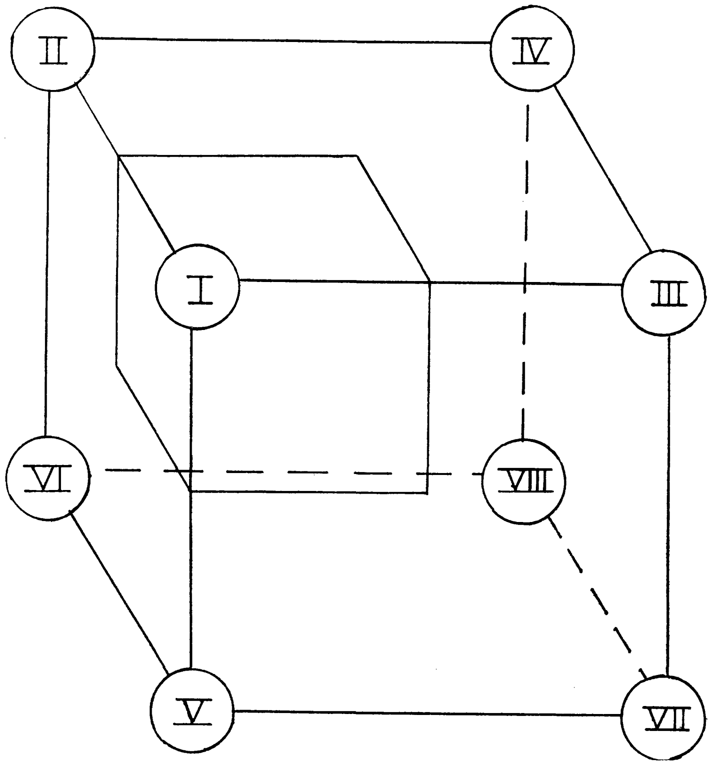


TABLE 1

POSITION	VISIBLE FACES
I	Top, Left, Front
II	Top, Left, Back
III	Top, Right, Front
IV	Top, Right, Back
V	Bottom, Left, Front
VI	Bottom, Left, Back
VII	Bottom, Right, Front
VIII	Bottom, Right, Back

To determine the number of distinct color configurations which the large cube can have, observe that the cube can always be oriented so that the white face is on the top. Then there are five different ways of choosing the bottom color. For each of these possibilities, the remaining four colors must fill four positions arranged in a circular pattern (-- left -- back -- right -- front --). The number of ways this can be done is $3! = 6$. Thus, there are $5 \times 6 = 30$ possible distinct color configurations for the large cube. (If one configuration can be obtained from another by rotation, then the two are considered to be identical.) The thirty possible cases are presented in Table 2.

TABLE 2

CASE	FACE:	TOP	BOTTOM	LEFT	RIGHT	FRONT	BACK
1		White	Yellow	Purple	Green	Red	Blue
2		White	Yellow	Purple	Green	Blue	Red
3		White	Yellow	Purple	Blue	Green	Red
4		White	Yellow	Purple	Blue	Red	Green
5		White	Yellow	Purple	Red	Blue	Green
6		White	Yellow	Purple	Red	Green	Blue
7		White	Purple	Yellow	Green	Blue	Red
8		White	Purple	Yellow	Green	Red	Blue
9		White	Purple	Yellow	Blue	Red	Green
10		White	Purple	Yellow	Blue	Green	Red
11		White	Purple	Yellow	Red	Blue	Green
12		White	Purple	Yellow	Red	Green	Blue
13		White	Red	Green	Blue	Purple	Yellow
14		White	Red	Green	Blue	Yellow	Purple
15		White	Red	Green	Yellow	Blue	Purple
16		White	Red	Green	Yellow	Purple	Blue
17		White	Red	Green	Purple	Blue	Yellow
18		White	Red	Green	Purple	Yellow	Blue
19		White	Green	Red	Blue	Purple	Yellow
20		White	Green	Red	Blue	Yello	Purple
21		White	Green	Red	Yellow	Bluew	Purple
22		White	Green	Red	Yellow	Purple	Blue
23		White	Green	Red	Purple	Blue	Yellow
24		White	Green	Red	Purple	Yellow	Blue
25		White	Blue	Red	Yellow	Green	Purple
26		White	Blue	Red	Yellow	Purple	Green
27		White	Blue	Red	Purple	Green	Yellow
28		White	Blue	Red	Purple	Yellow	Green
29		White	Blue	Red	Green	Yellow	Purple
30		White	Blue	Red	Green	Purple	Yellow

Associate a number with each of the six colors, letting 1 correspond to red, 2 to white, 3 to blue, 4 to green, 5 to purple, and 6 to yellow. Next, we define the external numbers for each position in the cube as follows:

Let a = number associated with the color on the top of the cube,
b = number associated with the color on the bottom of the cube,
c = number associated with the color on the left of the cube,
d = number associated with the color on the right of the cube,
e = number associated with the color on the front of the cube,
f = number associated with the color on the back of the cube,

Then Table 3 gives the formulas for computing the external numbers:

TABLE 3

POSITION	FORMULA FOR EXTERNAL NUMBER	POSITION	FORMULA FOR EXTERNAL NUMBER
I	$100a + 10c + e$	V	$100b + 10c + e$
II	$100a + 10c + f$	VI	$100b + 10c + f$
III	$100a + 10d + e$	VII	$100b + 10d + e$
IV	$100a + 10d + f$	VIII	$100b + 10d + f$

We now compute the external numbers corresponding to the thirty cases described in Table 2, thus obtaining Table 4:

TABLE 4

CASE	POSITION: I	II	III	IV	V	VI	VII	VIII
1	251	253	241	243	651	653	641	643
2	253	251	243	241	653	651	643	641
3	254	251	234	231	654	651	634	631
4	251	254	231	234	651	654	631	634
5	253	254	213	214	653	654	613	614
6	254	253	214	213	654	653	614	613
7	263	261	243	241	563	561	543	541
8	261	263	241	243	561	563	541	543
9	261	264	231	234	561	564	531	534
10	264	261	234	231	564	561	534	531
11	263	264	213	214	563	564	513	514
12	264	263	214	213	564	563	514	513
13	245	246	235	236	145	146	135	136
14	246	245	236	235	146	145	136	135
15	243	245	263	265	143	145	163	165
16	245	243	265	263	145	143	165	163
17	243	246	253	256	143	146	153	156
18	246	243	256	253	146	143	156	153
19	215	216	235	236	415	416	435	436
20	216	215	236	235	416	415	436	435
21	213	215	263	265	413	415	463	465
22	215	213	265	263	415	413	465	463
23	213	216	253	256	413	416	453	456
24	216	213	256	253	416	413	456	453
25	214	215	264	265	314	315	364	365
26	215	214	265	264	315	314	365	364
27	214	216	254	256	314	316	354	356
28	216	214	256	254	316	314	356	354
29	216	215	246	245	316	315	346	345
30	215	216	245	246	315	316	345	346

For each of the smaller cubes, determine the external numbers which are possible for each position in the large cube. These are listed in Table 5:

TABLE 5

CUBE 1

POSITION	POSSIBLE EXTERNAL NUMBERS										
I	216,	235,	251,	263								
II	215,	236,	253,	261								
III	215,	236,	253,	261								
IV	216,	235,	251,	263								
V	145,	164,	346,	354,	416,	435,	451,	463,	514,	543,	634,	641
VI	146,	154,	345,	364,	415,	436,	453,	461,	534,	541,	614,	643
VII	146,	154,	345,	364,	415,	436,	453,	461,	534,	541,	614,	643
VIII	145,	164,	346,	354,	416,	435,	451,	463,	514,	543,	634,	641

CUBE 2

POSITION	POSSIBLE EXTERNAL NUMBERS											
I	235,	243,	256,	264								
II	234,	246,	253,	265								
III	234,	246,	253,	265								
IV	235,	243,	256,	264								
V	135,	143,	156,	164,	314,	351,	416,	431,	513,	561,	615,	641
VI	134,	146,	153,	165,	315,	341,	413,	461,	516,	531,	614,	651
VII	134,	146,	153,	165,	315,	341,	413,	461,	516,	531,	614,	651
VIII	135,	143,	156,	164,	314,	351,	416,	431,	513,	561,	615,	641

CUBE 3

POSITION	POSSIBLE EXTERNAL NUMBERS											
I	214,	246,	251,	265								
II	215,	241,	256,	264								
III	215,	241,	256,	264								
IV	214,	246,	251,	265								
V	135,	143,	314,	346,	351,	365,	431,	463,	513,	536,	634,	653
VI	134,	153,	315,	341,	356,	364,	413,	436,	531,	563,	635,	643
VII	134,	153,	315,	341,	356,	364,	413,	436,	531,	563,	635,	643
VIII	135,	143,	314,	346,	351,	365,	431,	463,	513,	536,	634,	653

TABLE 5 — continued

CUBE 4

POSITION	POSSIBLE EXTERNAL NUMBERS										
I	214,	236,	243,	261							
II	216,	234,	241,	263							
III	216,	234,	241,	263							
IV	214,	236,	243,	261							
V	145,	156,	354,	365,	435,	451,	514,	536,	543,	561,	615,	653
VI	154,	165,	345,	356,	415,	453,	516,	534,	541,	563,	635,	651
VII	154,	165,	345,	356,	415,	453,	516,	534,	541,	563,	635,	651
VIII	145,	156,	354,	365,	435,	451,	514,	536,	543,	561,	615,	653

CUBE 5

POSITION	POSSIBLE EXTERNAL NUMBERS											
I	236,	243,	254,	265								
II	234,	245,	256,	263								
III	234,	245,	256,	263								
IV	236,	243,	254,	265								
V	136,	143,	154,	165,	314,	361,	415,	431,	516,	541,	613,	651
VI	134,	145,	156,	163,	316,	341,	413,	451,	514,	561,	615,	631
VII	134,	145,	156,	163,	316,	341,	413,	451,	514,	561,	615,	631
VIII	136,	143,	154,	165,	314,	361,	415,	431,	516,	541,	613,	651

CUBE 6

POSITION	POSSIBLE EXTERNAL NUMBERS											
I	214,	231,	246,	263								
II	213,	236,	241,	264								
III	213,	236,	241,	264								
IV	214,	231,	246,	263								
V	145,	153,	315,	356,	451,	465,	514,	531,	546,	563,	635,	654
VI	135,	154,	351,	365,	415,	456,	513,	536,	541,	564,	645,	653
VII	135,	154,	351,	365,	415,	456,	513,	536,	541,	564,	645,	653
VIII	145,	153,	315,	356,	451,	465,	514,	531,	546,	563,	635,	654

TABLE 5 — continued

CUBE 7

POSITION	POSSIBLE EXTERNAL NUMBERS										
I	213,	235,	256,	261							
II	216,	231,	253,	265							
III	216,	231,	253,	265							
IV	213,	235,	256,	261							
V	134,	146,	341,	354,	413,	435,	456,	461,	543,	564,	614,	645
VI	143,	164,	314,	345,	416,	431,	453,	465,	534,	546,	641,	654
VII	143,	164,	314,	345,	416,	431,	453,	465,	534,	546,	641,	654
VIII	134,	146,	341,	354,	413,	435,	456,	461,	543,	564,	614,	645

CUBE 8

POSITION	POSSIBLE EXTERNAL NUMBERS											
I	216,	245,	251,	264								
II	215,	246,	254,	261								
III	215,	246,	254,	261								
IV	216,	245,	251,	264								
V	135,	163,	316,	345,	351,	364,	436,	453,	513,	534,	631,	643
VI	136,	153,	315,	346,	354,	361,	435,	463,	531,	543,	613,	634
VII	136,	153,	315,	346,	354,	361,	435,	463,	531,	543,	613,	634
VIII	135,	163,	316,	345,	351,	364,	436,	453,	513,	534,	631,	643

We now compare Tables 4 and 5 to determine which, if any, of the thirty possible cases can be eliminated from consideration as a solution. A case will be eliminated if for at least one of the eight small cubes, it is true that none of the possible external numbers of that cube (Table 5) correspond with any of the external numbers of the case in question (Table 4). This comparison yields Table 6, which tells what cases are eliminated by examination of specific cubes:

TABLE 6

CUBE	ELIMINATED CASES								
1	3,	5,	9,	12,	15,	18,	19,	26,	27
2	4,	5,	7,	11,	18,	19,	21,	28,	30
3	3,	6,	7,	10,	13,	16,	19,	23,	26
4	4,	6,	7,	14,	18,	22,	24,	26,	29
5	2,	5,	9,	11,	16,	20,	23,	26,	29
6	2,	4,	12,	15,	17,	19,	23,	28,	29
7	2,	3,	7,	11,	13,	15,	24,	25,	29
8	2,	6,	8,	11,	15,	18,	22,	23,	30

Interpreting the data of Table 6, we conclude that the only case not eliminated as a possible solution is the first. Comparing the external numbers for Case 1 with the information in Table 5, it follows that each small cube can conceivably be located in two positions:

TABLE 7

CUBE	POSITION
1	I, II
2	II, IV
3	I, III
4	III, IV
5	(IV), V
6	(III), VI
7	(II), VII
8	(I), VIII

It is obvious that in a given solution, none of the small cubes can occupy a position enclosed in parentheses in Table 7. Hence, cubes 5–8 must be located in positions V–VIII respectively. If Cube 1 is in position I, then Cube 3 must be in position III, Cube 4 in position IV, and Cube 2 in position II; if Cube 1 is in position II, then Cube 2 must be in position IV, Cube 4 in position III, and Cube 3 in position I. Therefore, there exist two solutions, both of which give the same color configuration in the large cube:

SOLUTION 1				SOLUTION 2			
CUBE	POSITION	CUBE	POSITION	CUBE	POSITION	CUBE	POSITION
1	I	5	V	1	II	5	V
2	II	6	VI	2	IV	6	VI
3	III	7	VII	3	I	7	VII
4	IV	8	VIII	4	III	8	VIII

It is now a simple matter to assemble the large cube in the desired way, since each cube is distinctly numbered, each position number indicates which three faces are to be visible (Table 1), and the color of each face of the large cube is uniquely determined (Table 2, Case 1).

Notes. 1. "Eight Blocks to Madness" was invented by Mr. Eric Cross of Ireland. It is manufactured and distributed by Austin Enterprises, 2108 Braewick Circle, Akron, Ohio 44313.

2. In Volume 2, Number 1 of the Journal of Recreational Mathematics, there is an article entitled *The Mayblox Problem*, written by Margaret A. Farrell, SUNY, Albany. This problem, while similar in nature to "Eight Blocks to Madness", requires an additional condition for solution; namely,

that the contact faces of the small cubes be the same color. On the surface this seems to present added difficulty, but in fact, it acts as a key for a method of solution virtually free of cumbersome tables. Unfortunately, it is impossible to solve the Mayblox problem given the specific cubes of Diagram 1. This impossibility is most easily seen in retrospect, upon examination of the two solutions found above. Hence, the somewhat simpler method used by Miss Farrell is not applicable to the present problem.

3. The compilation of the tables involves a few hours of rather tedious labor, but this should not be a deterrent to the avid puzzle solver. While it is certainly possible to discover the solution in a shorter period of time by chance, such a course affords no insurance that the feat can be repeated once the cubes are again mixed. The solution presented here provides the guarantee that the puzzle can be solved within a minute, once the tables have been computed. As a byproduct, the solution indicates that the color scheme of the large cube is unique up to orientation.

ON CERTAIN CHAINS

S. S. SUBRAMANYAM, Tirupati, India

1. Introduction. D. W. Babbage [1] has considered a chain of points and equal circles, using the circular coordinates of F. Morley and F. V. Morley [7]. There are interesting chains consisting of lines, points and equal circles, concerning the well-known θ-pedal lines of triads. We consider a few such chains in the present paper using the circular coordinate system (see W. B. Carver [2] for details of this coordinate system).

A base circle will be assumed to be given in all the cases, and will be chosen as the unit circle for the coordinate system. It is known [8] that the θ-pedal line of the point τ with respect to the triad $t_1t_2t_3$ ($|\tau| = 1 = |t_i|$) on the unit circle has self-conjugate equation

$$(1.1) \qquad \frac{\gamma\tau}{t_1t_2t_3}x + y = \frac{1}{\tau} - \gamma\left[\prod_1^3 (\tau - t_j)\right]/(1-\gamma)t_1t_2t_3\tau + \frac{\gamma\tau^2}{t_1t_2t_3}$$

($\gamma = e^{-2i\theta}$) and the θ-deltoid (envelope of θ-pedal lines) of the triad $t_1t_2t_3$ has map equation [9]

$$(1.2) \qquad x = \frac{\Sigma_1}{1-\gamma} - \frac{\gamma}{1-\gamma}T^2(2t + 1/t^2) \quad (\Sigma_1 = t_1 + t_2 + t_3;\ T^6 = t_1t_2t_3/\gamma^2).$$

2. θ-chain for fixed τ on base circle. Having obtained the θ-pedal line L^{123} of τ with respect to the triad $t_1t_2t_3$, if we take another point t_4 on the base circle, the four θ-pedal lines of τ with respect to the four triads contained in the tetrad $t_1t_2t_3t_4$ are

$$(2.1) \qquad \begin{aligned} \frac{\gamma\tau t_j}{\Sigma_4}x + y &= \gamma\tau t_j\left(\Sigma_1 - \gamma\tau - 2t_j - \frac{\Sigma_4}{\tau t_j^2}\right)/(1-\gamma)\Sigma_4 \\ &\quad + \frac{\gamma}{\gamma - 1}\left(\frac{\Sigma_3}{\Sigma_4} - \frac{1}{\gamma\tau} - \frac{2}{t_j} - \frac{\tau t_j^2}{\Sigma_4}\right) \end{aligned}$$

that the contact faces of the small cubes be the same color. On the surface this seems to present added difficulty, but in fact, it acts as a key for a method of solution virtually free of cumbersome tables. Unfortunately, it is impossible to solve the Mayblox problem given the specific cubes of Diagram 1. This impossibility is most easily seen in retrospect, upon examination of the two solutions found above. Hence, the somewhat simpler method used by Miss Farrell is not applicable to the present problem.

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($\gamma = e^{-2i\theta}$) and the θ-deltoid (envelope of θ-pedal lines) of the triad $t_1t_2t_3$ has map equation [9]

$$(1.2)\qquad x = \frac{\Sigma_1}{1-\gamma} - \frac{\gamma}{1-\gamma}T^2(2t + 1/t^2) \quad (\Sigma_1 = t_1 + t_2 + t_3;\ T^6 = t_1t_2t_3/\gamma^2).$$

2. θ-chain for fixed τ on base circle. Having obtained the θ-pedal line L^{123} of τ with respect to the triad $t_1t_2t_3$, if we take another point t_4 on the base circle, the four θ-pedal lines of τ with respect to the four triads contained in the tetrad $t_1t_2t_3t_4$ are

$$(2.1)\qquad \begin{aligned}\frac{\gamma\tau t_j}{\Sigma_4}x + y &= \gamma\tau t_j\left(\Sigma_1 - \gamma\tau - 2t_j - \frac{\Sigma_4}{\tau t_j^2}\right)/(1-\gamma)\Sigma_4 \\ &\quad + \frac{\gamma}{\gamma-1}\left(\frac{\Sigma_3}{\Sigma_4} - \frac{1}{\gamma\tau} - \frac{2}{t_j} - \frac{\tau t_j^2}{\Sigma_4}\right)\end{aligned}$$

($j = 1, 2, 3, 4$; Σ's the elementary symmetric functions of the four t's). The form shows that these four lines touch a deltoid D^{1234} with map equation

$$x = (\Sigma_1 - \gamma\tau)/(1-\gamma) - \frac{1}{1-\gamma}(\Sigma_4/\tau)^{1/3}(2t + 1/t^2) \tag{2.2}$$

(Σ's for four t's), at the points $t = (\tau/\Sigma_4)^{1/3}t_j$. This deltoid is equal in size to the θ-deltoid of any of the four triads, and has center O^{1234} at the point $x = (\Sigma_1 - \gamma\tau)/(1-\gamma)$.

Choosing another point t_5 on the base circle, we have five deltoids corresponding to the five tetrads contained in the pentad $t_1t_2\cdots t_5$, all equal in size. Their centers lie on the circle $(O^{12\cdots 5})$ with map equation

$$x = (\Sigma_1 - \gamma\tau)/(1-\gamma) + \frac{1}{1-\gamma}t \tag{2.3}$$

($\Sigma_1 = t_1 + \cdots + t_5$),center $O^{12\cdots 5}$ at $x = (t_1 + \cdots + t_5 - \gamma\tau)/(1-\gamma)$ and equal to the inscribed circle of any of the five deltoids, its radius being independent of τ.

With another point t_6 on the base circle, we will get six equal circles like the $(O^{12\cdots 5})$ above, having their centers on another equal circle $(O^{12\cdots 6})$ with map equation

$$x = (\Sigma_1 - \gamma\tau)/(1-\gamma) + \frac{1}{1-\gamma}t \tag{2.4}$$

($\Sigma_1 = t_1 + \cdots + t_6$), so that these six circles have a common point $O^{12\ldots 6}$, namely the center of the circle $(O^{12\cdots 6})$.

Proceeding thus, we have at any stage a circle $(O^{12\cdots j})$, $j > 4$, with its center $O^{12\cdots j}$ as a common point for j equal circles whose centers lie on $(O^{12\cdots j})$. The next stage gives a circle $(O^{12\cdots j+1})$ containing $j+1$ points like the $O^{12\cdots j}$ of the previous stage, and its center $O^{12\cdots j+1}$ is common to $j+1$ circles like the $(O^{12\cdots j})$. This chain thus starts with four θ-pedal lines and a deltoid determined by them and consists successively of five equal deltoids with centers on a circle, six equal circles having a common point and with centers on an equal circle, and so on:

$$(t_1, t_2, \cdots; \tau) \Rightarrow L^{123}, L^{234}, L^{134}, L^{124}; D^{1234} \Rightarrow (O^{12\ldots 5}) \Rightarrow (O^{12\ldots 6}) \Rightarrow \cdots.$$

3. θ-chain for varying τ on base circle. Starting with a tetrad $t_1t_2t_3t_4$ on the base circle, the four θ-pedal lines of t_p with respect to the triads $t_qt_rt_s$ ($p, q, r, s = 1,2,3,4$) are

$$\frac{\gamma t_j^2}{\Sigma_4}x + y = \frac{\gamma\Sigma_1 t_j^2}{(1-\gamma)\Sigma_4} - \frac{\gamma(1+\gamma)}{(1-\gamma)\Sigma_4}t_j^3 + \frac{1+\gamma}{1-\gamma}\cdot\frac{1}{t_j} - \frac{\gamma\Sigma_3}{(1-\gamma)\Sigma_4} \tag{3.1}$$

($j = 1, 2, 3, 4$; Σ's for the four t's). These four lines obviously touch a curve with tangential equation

$$\frac{\gamma t^2}{\Sigma_4}x + y = \frac{\gamma\Sigma_1 t^2}{(1-\gamma)\Sigma_4} - \frac{\gamma(1+\gamma)}{(1-\gamma)\Sigma_4}t^3 + \frac{1+\gamma}{1-\gamma}\cdot\frac{1}{t} - \frac{\gamma\Sigma_3}{(1-\gamma)\Sigma_4} \tag{3.2}$$

and its map equation is

$$x = \frac{\Sigma_1}{1-\gamma} - \frac{1}{2}\cdot\frac{1+\gamma}{1-\gamma}(\Sigma_4/\gamma)^{\frac{1}{4}}(3t + 1/t^3) \tag{3.3}$$

representing a four-cusped hypocycloid [2], also called a regular astroid, with center O^4 at $x = (t_1 + t_2 + t_3 + t_4)/(1-\gamma)$ and whose inscribed circle has its radius equal to cot θ times that of the base circle.

Choosing one more point t_5 on the base circle, we have five equal astroids whose centers lie on a circle (O^5) with map equation

$$x = \frac{\Sigma_1}{1-\gamma} + \frac{1}{1-\gamma}t \tag{3.4}$$

$(\Sigma_1 = t_1 + \cdots + t_5)$, center O^5 at $x = (t_1 + t_2 + \cdots + t_5)/(1-\gamma)$ and which is equal in size to the inscribed circle of the θ-deltoid of any triad on the base circle.

Proceeding in similar manner, the rest is as in Section 2. This chain thus starts with four θ-pedal lines touching a regular astroid and consists successively of five equal astroids with centers on a circle, six equal circles having a common point and with centers on another equal circle, and so on:

$$(t_1, t_2, \cdots) \Rightarrow L_1^{234}, L_2^{134}, L_3^{124}, L_4^{123}; H^4 \Rightarrow (O^5) \Rightarrow (O^6) \Rightarrow \cdots.$$

4. Simson chain. For a given tetrad $t_1t_2t_3t_4$ on the base circle, the Simson lines ($\pi/2$ -pedal lines) S_p^{qrs} of t_p with respect to the four triads $t_qt_rt_s$ $(p,q,r,s = 1,2,3,4)$ are all concurrent at the point $x = (t_1 + t_2 + t_3 + t_4)/2$, as can be seen from equation (3.1); this point of concurrence is the common midpoint of the joins of t_p to the orthocenters H_p of the triads $t_qt_rt_s$, and we may call it the Simson point S^4 of the tetrad $t_1t_2t_3t_4$. S. R. Mandan [5] has shown that the tetrad $H_1H_2H_3H_4$ formed by the orthocenters of the four triads contained in $t_1t_2t_3t_4$ is congruent to the latter, and each tetrad is obtainable from the other in the same way; two such tetrads have been called conjugate tetrads. The nine-point centers of the four triads contained in either tetrad have been shown to lie on the same circle, called the M-circle of either tetrad. This common circle has, in fact, the map equation

$$x = \tfrac{1}{2}(t_1 + t_2 + t_3 + t_4) + \tfrac{1}{2}t \tag{4.1}$$

and the center of this circle is precisely the Simson point S^4 of the tetrad; this shows, incidentally, that *the Simson points of conjugate tetrads are coincident.*

Choosing another point t_5 on the base circle, the Simson lines of t_4 and t_5 with respect to the triad $t_1t_2t_3$ meet in the point

$$x = \tfrac{1}{2}(t_1 + t_2 + \cdots + t_5) + \tfrac{1}{2}t_1t_2t_3/t_4t_5$$

so that the $\binom{5}{2} = 10$ such points lie on a circle with map equation

$$x = \tfrac{1}{2}(t_1 + t_2 + \cdots + t_5) + \tfrac{1}{2}t. \tag{4.2}$$

This circle contains also the five Simson points of the five tetrads contained in the pentad $t_1t_2\cdots t_5$; it may hence be called *the Simson circle* (S^5) *of the pentad*, and is precisely the M-circle of the pentad in [5]; its center S^5 may be called *the Simson point of the pentad.*

With one more point t_6 on the base circle, the Simson circles of the six pentads contained in the hexad $t_1t_2\cdots t_6$ are all equal and have their centers on a circle (S^6) with map equation

$$x = \tfrac{1}{2}(t_1 + t_2 + \cdots + t_6) + \tfrac{1}{2}t \tag{4.3}$$

whose center S^6 is at the point $x = \frac{1}{2}(t_1 + t_2 + \cdots + t_6)$ and is the common point on the six equal circles. This point S^6, which we may call the *Simson point of the hexad,* is the common midpoint of the 20 joins of the orthocenters of triads $t_lt_mt_n$ to those of $t_pt_qt_r$ $(l, m, n, p, q, r = 1, 2, \cdots, 6)$. It may be seen that this point S^6 is also the common center of the 20 circumcircles of triangles formed by the Simson lines of three vertices of the hexad with respect to the remaining triad; for the circumcircle of the triangle formed by the Simson lines of t_4, t_5, t_6 with respect to $t_1t_2t_3$, has map equation

$$x = \tfrac{1}{2}(t_1 + t_2 + \cdots + t_6) + \tfrac{1}{2}\left(\frac{t_1t_2t_3}{t_4t_5t_6} - 1\right)t. \tag{4.4}$$

Thus, we have a chain, which we may call the *Simson chain,* starting with four concurrent Simson lines and consisting successively of five such points of concurrence lying on a circle, six equal circles having a common point and with their centers lying on another equal circle, and so on:

$$(t_1, t_2, \cdots) \Rightarrow S_1^{234}, S_2^{134}, S_3^{124}, S_4^{123};\ S^4 \Rightarrow (S^5) \Rightarrow (S^6) \Rightarrow \cdots.$$

5. Limiting cases of θ-Kantor chain. M. Kobayashi [4] proved a theorem due to Kantor which establishes a chain concerning Simson lines. Chi-Ho-Loong [3] and T. Minoda [6] developed a similar chain for θ-pedal lines; this may be called the *θ-Kantor chain.* We shall now obtain a few interesting limiting cases of this θ-Kantor chain.

Start with two distinct sets of points $t_1, t_2, t_3, \cdots; \tau_1, \tau_2, \cdots$ on the base circle, the t-set containing not less than three points and the τ-set not less than two points. Let Σ's and σ's denote the elementary symmetric functions of the t's and the τ's respectively.

Denote by L_1^{123} the θ-pedal line of τ_1 with respect to the triad $t_1t_2t_3$ given by equation (1.1) above. Now, the θ-pedal lines of τ_1, τ_2 with respect to $t_1t_2t_3$ meet in the point O_{12}^{123} given by

$$x = \frac{\Sigma_1}{1-\gamma} - \frac{\gamma}{1-\gamma}(\tau_1 + \tau_2) - \frac{\Sigma_3}{\gamma(1-\gamma)} \cdot \frac{1}{\tau_1\tau_2}$$

(Σ's for the three t's).

Next, four points t_1, t_2, t_3, t_4 of the t-set give rise to the four points O_{12}^{123}, O_{12}^{234}, O_{12}^{134}, O_{12}^{124} which lie on a line L_{12}^{1234} with self-conjugate equation

$$(5.1)\ \frac{\gamma^2\tau_1\tau_2}{\Sigma_4}x + y = \frac{\gamma^2\tau_1\tau_2}{(1-\gamma)\Sigma_4}\Sigma_1 - \frac{\gamma^3\tau_1\tau_2}{(1-\gamma)\Sigma_4}(\tau_1+\tau_2) + \frac{\gamma\Sigma_3}{(\gamma-1)\Sigma_4} - \frac{(\tau_1+\tau_2)}{(\gamma-1)\tau_1\tau_2}$$

(Σ's for the four t's).

Again, for three points τ_1, τ_2, τ_3 of the τ-set, the three lines L_{12}^{1234}, L_{23}^{1234}, L_{13}^{1234} meet at the point O_{123}^{1234} given by

$$x = (\Sigma_1 - \gamma\sigma_1)/(1-\gamma) - \frac{1}{1-\gamma}\cdot\frac{\Sigma_4}{\gamma^2\sigma_3}$$

(Σ's for the four t's; σ's for the three τ's).

This process, continued indefinitely by choosing one more point each time alternately from the two sets, yields a chain of lines of the form $L_{12\ldots n-2}^{12\cdots n}$ given by

$$(5.2)\quad \frac{\gamma^{n-2}\sigma_{n-2}}{\Sigma_n}x + y = \frac{\gamma^{n-2}\sigma_{n-2}\Sigma_1}{(1-\gamma)\Sigma_n} - \frac{\gamma^{n-1}\sigma_{n-2}\sigma_1}{(1-\gamma)\Sigma_n} + \frac{\gamma\Sigma_{n-1}}{(\gamma-1)\Sigma_n} - \frac{\sigma_{n-3}}{(\gamma-1)\sigma_{n-2}}$$

(Σ's for n t's; σ's for $n-2$ τ's) and points of the form $O_{12\ldots n-1}^{12\cdots n}$ given by $x = (\Sigma_1 - \gamma\sigma_1)/(1-\gamma) - \dfrac{1}{1-\gamma}\cdot\dfrac{\Sigma_n}{\gamma^{n-2}\sigma_{n-1}}$ (Σ's for n t's; σ's for $n-1$ τ's). This is precisely the θ-Kantor chain:

$$(t_1, t_2, t_3, \cdots; \tau_1, \tau_2, \cdots) \Rightarrow L_1^{123} \Rightarrow O_{12}^{123} \Rightarrow L_{12}^{1234} \Rightarrow \cdots \Rightarrow L_{12\ldots n-2}^{12\ldots n} \Rightarrow O_{12\ldots n-1}^{12\ldots n} \Rightarrow \cdots.$$

We shall now allow some of the τ's to take coincident positions while the t's are all assumed to be distinct. As $\tau_2 \to \tau_1$, the point O_{12}^{123} becomes, in the limit, the point of contact O_{11}^{123} of L_1^{123} with the θ-deltoid D^{123} of the triad $t_1t_2t_3$, as we could only expect. The line L_{11}^{1234} with self-conjugate equation

$$(5.3)\quad \frac{\gamma^2\tau_1^2}{\Sigma_4}x + y = \frac{\gamma^2\tau_1^2\Sigma_1}{(1-\gamma)\Sigma_4} - \frac{2\gamma^3\tau_1^3}{(1-\gamma)\Sigma_4} + \frac{\gamma\Sigma_3}{(\gamma-1)\Sigma_4} - \frac{2}{(\gamma-1)\tau_1}$$

(Σ's for the four t's) is tangent at the point corresponding with $t = (\gamma^3/\Sigma_4)^{\frac{1}{4}}\tau_1$ to the regular astroid H^{1234} with map equation

$$(5.4)\quad x = \frac{\Sigma_1}{1-\gamma} - \frac{\gamma}{1-\gamma}(\Sigma_4/\gamma^3)^{\frac{1}{4}}(3t + 1/t^3).$$

This regular astroid for the tetrad may be considered the direct analogue of the θ-deltoid for a triad.

In the next stage, if $\tau_2, \tau_3 \to \tau_1$, the point O_{111}^{1234} is precisely the point of contact of L_{11}^{1234} with the astroid of the previous stage. In the next stage, five such points are collinear on the line $L_{111}^{12\ldots5}$ which touches a five cusped hypocycloid at the point the point $O_{1111}^{12\ldots5}$; and so on.

Thus, in this limiting process, the θ-Kantor chain reduces to a chain consisting of lines and their points of contact with certain regular hypocycloids; for instance, the line $L_{11\ldots1}^{12\cdots n}$ ($n-2$ subscripts) with self-conjugate equation

$$(5.5)\ \frac{\gamma^{n-2}\tau_1^{n-2}}{\Sigma_n}x + y = \frac{\gamma^{n-2}\tau_1^{n-2}\Sigma_1}{(1-\gamma)\Sigma_n} - (n-2)\frac{\gamma^{n-1}\tau_1^{n-1}}{(1-\gamma)\Sigma_n} + \frac{\gamma\Sigma_{n-1}}{(\gamma-1)\Sigma_n} - (n-2)\frac{1}{(\gamma-1)\tau_1}$$

(Σ's for n t's) touches the n-cusped hypocycloid $H^{12\cdots n}$ given by

$$x = \frac{\Sigma_1}{1-\gamma} - \frac{\gamma}{(1-\gamma)}(\Sigma_n/\gamma^{n-1})^{1/n}[(n-1)t + 1/t^{n-1}] \tag{5.6}$$

at the point $O_{11\cdots 1}^{12\cdots n}$ ($n-1$ subscripts) corresponding to $t = (\gamma^{n-1}/\Sigma_n)^{1/n}\tau_1$ on the curve.

$$(t_1, t_2, \cdots; \tau_1 = \tau_2 = \cdots) \Rightarrow L_1^{123}; D^{123}; O_{11}^{123} \Rightarrow L_{11}^{1234}; H^{1234};$$
$$O_{111}^{1234} \Rightarrow \cdots \Rightarrow L_{11\ldots 1}^{12\ldots n}; H^{12\ldots n}; O_{11\ldots 1}^{12\ldots n} \Rightarrow .$$

6. Branch chains. Besides the main θ-Kantor chain obtained as above, we have at each stage a branch chain:

(i) After obtaining the line L_1^{123}, if instead of choosing one more point τ_2 from the τ-set, we choose a point t_4 from the t-set, the four lines like L_1^{123} touch a deltoid D_1^{1234} and, continuing this process of choosing one more point every time from the t-set only, we will have the chain obtained in Section 2.

(ii) Having obtained the point O_{12}^{123}, if we continue to choose points from the τ-set only, we will have in the next stage a circle (O_{123}^{123}), namely the circumcircle of the triangle formed by the θ-pedal lines of τ_1, τ_2, τ_3 with respect to $t_1t_2t_3$, with map equation

$$x = (\Sigma_1 - \gamma\sigma_1)/(1-\gamma) - \left[\frac{\Sigma_3}{\gamma(1-\gamma)\sigma_3} - \frac{\gamma}{1-\gamma}\right]t \tag{6.1}$$

(Σ's for three t's; σ's for three τ's). In the next stage, we will have four equal circles like the one above, their centers lying on another circle (O_{1234}^{123}) with map equation

$$x = (\Sigma_1 - \gamma\sigma_1)/(1-\gamma) + \frac{\gamma}{1-\gamma}t \tag{6.2}$$

($\Sigma_1 = t_1 + t_2 + t_3$; $\sigma_1 = \tau_1 + \tau_2 + \tau_3 + \tau_4$). Thereafter, we get a sequence of equal circles with a common point and their centers lying on another equal circle, as in Section 2.

(iii) Again, if after obtaining the line L_{12}^{1234} we choose points from the t-set only we will have in the next stage five lines like L_{12}^{1234} touching a deltoid $D_{12}^{12\ldots 5}$ with map equation

$$x = [\Sigma_1 - \gamma(\tau_1 + \tau_2)]/(1-\gamma) - \frac{1}{1-\gamma}(\Sigma_5/\gamma\tau_1\tau_2)^{\frac{1}{3}}(2t + 1/t^2) \tag{6.3}$$

(Σ's for five t's). The next stage will give six such equal deltoids with centers lying on a circle ($O_{12}^{12\ldots 6}$) with map equation

$$x = [\Sigma_1 - \gamma(\tau_1 + \tau_2)]/(1-\gamma) + \frac{1}{1-\gamma}t \tag{6.4}$$

($\Sigma_1 = t_1 + \cdots + t_6$), whose radius is independent of the t's. Hence, we get again a chain of points and equal circles, as in Section 2.

There are thus two types of branch chains for the main θ-Kantor chain: one type starts with a line of the form $L^{12\dots n}_{12\dots n-2}$ given by equation (5.2) and consists successively of $n+1$ such lines touching the deltoid $D^{12\dots n+1}_{12\dots n-2}$ given by

$$x = (\Sigma_1 - \gamma\sigma_1)/(1-\gamma) - \frac{1}{1-\gamma}(\Sigma_{n+1}/\gamma^{n-3}\sigma_{n-2})^{1/3}(2t + 1/t^2) \tag{6.5}$$

(Σ's for $n+1$ t's; σ's for $n-2$ τ's), the centers of $n+2$ such equal deltoids lying on a circle $(O^{12\dots n+2}_{12\dots n-2})$ with map equation

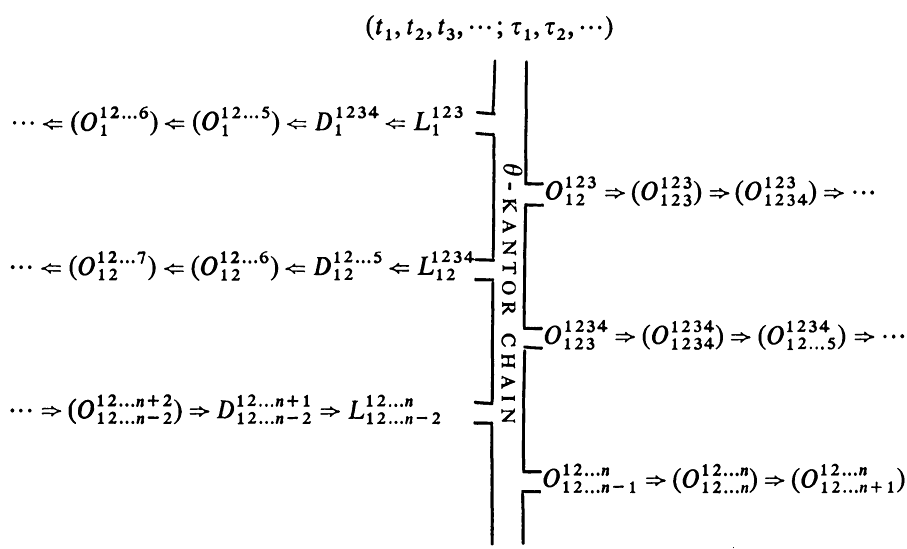
$$x = (\Sigma_1 - \gamma\sigma_1)/(1-\gamma) + \frac{1}{1-\gamma}t \tag{6.6}$$

($\Sigma_1 = t_1 + \cdots + t_{n+2}$; $\sigma_1 = \tau_1 + \cdots + \tau_{n-2}$), $n+3$ such equal circles having a common point and their centers lying on another equal circle, and so on; the other type starts with a point of the form $O^{12\dots n}_{12\dots n-1}$, n such points lying on a circle $(O^{12\dots n}_{12\dots n})$ with map equation

$$x = (\Sigma_1 - \gamma\sigma_1)/(1-\gamma) - \left[\frac{\Sigma_n}{(1-\gamma)\gamma^{n-2}\sigma_n} - \frac{\gamma}{1-\gamma}\right]t \tag{6.7}$$

(Σ's for n t's; σ's for n τ's), $n+1$ such equal circles having a common point and their centers lying on another equal circle, and so on. These two types are shown in the tree-diagram at the end.

The circle $(O^{12\dots n}_{12\dots n})$ above is seen to be unaltered if we change γ into $1/\gamma$ and at the same time interchange the Σ's and the σ's. Thus, if we call it the θ-circle of the n t's and the n τ's, we see that it is also the $(\pi - \theta)$-circle of the n τ's and the n t's. Again, if $\theta = \pi/2$, the similar circles that we would have on partitioning in all possible ways $2n$ given cyclic points into two sets of n each, would all be concentric. The θ-circle above has been called by T. Minoda [6] the Kantor circle, for amplitude θ, of n t's and n τ's,



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CONTINUOUS MULTIPLICATIONS IN R^2

G. A. HEUER, Concordia College

1. Introduction. In the real numbers R with the usual addition, it is rather well known that any nontrivial continuous multiplication which results in an associative ring results in one isomorphic to the usual real number field. (If $1*1 = a$, then $x*y = xya$, and if $a \neq 0$, the map $x \mapsto ax$ is an isomorphism onto R with usual multiplication.)

The direct sum $R \oplus R$ and the complex number field C are two familiar examples of rings in which the additive group is R^2 under componentwise addition. What other multiplications are there in R^2 such that the resulting system is a ring? ("Ring" will always mean associative ring.) We shall see that if multiplication is continuous then there are exactly eight isomorphism classes, of which six are commutative. Three (all commutative) have identity elements; the two noncommutative classes are anti-isomorphic. We shall give abstract characterizations of each of these eight rings, as well as representations as matrix algebras over R.

The analysis is applicable somewhat more generally. The eight multiplications are available in general, but may not yield eight distinct isomorphism classes, and may not yield all the isomorphism classes. If R is any subdomain of the real numbers or any of the rings $Z/(k)$, where Z is the ring of integers, then the eight classes are distinct. If R is a subfield of the reals or one of the fields $Z/(p)$, p a prime, then there are precisely these eight isomorphism classes (still assuming continuity of multiplication in the former case).

Fuchs [1] discusses the general question of what rings exist having a given additive group, but does not enumerate the isomorphism classes and characterize them abstractly for the cases we are considering. Rather than derive our results from his work, we shall obtain them directly from entirely elementary considerations.

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Fuchs [1] discusses the general question of what rings exist having a given additive group, but does not enumerate the isomorphism classes and characterize them abstractly for the cases we are considering. Rather than derive our results from his work, we shall obtain them directly from entirely elementary considerations.

2. The bilinearity conditions. Let $S = R^2$, with $(a,b)+(c,d)=(a+c,b+d)$ and $(a,b)(c,d)=(f_1(a,b,c,d),\ f_2(a,b,c,d))$. What functions f_1, f_2 are there for which this system is a ring? For r in R, $r(a,b)$ will mean (ra,rb) as usual, and we may consider R^2 as a vector space over R.

THEOREM 2.1. *Suppose S as described above is a ring. If r and s are rational, then* $f_i(r(a,b), s(c,d)) = rsf_i(a,b,c,d)$, *for* $i = 1,2$. *If the* f_i *are continuous, the same holds for arbitrary* r,s *in* R. (*Otherwise stated,* $r(a,b)s(c,d) = rs(a,b)(c,d)$.)

Proof. The first statement is easily established for r,s integers, using the distributive laws. It follows that for nonzero integers n,q, $nqf_i(a/n,b/n,c/q,d/q) = f_i(a,b,c,d)$, and thus if m,p are any integers, $f_i((m/n)(a,b),\ (p/q)(c,d)) = f_i(m(a/n,b/n),\ p(c/q,d/q)) = mpf_i(a/n,b/n,c/q,d/q) = (mp/nq)f_i(a,b,c,d)$. The last statement of the theorem is an immediate consequence of the density of the rationals in the reals.

It is only in the above theorem that the continuity of multiplication is used.

3. The associativity conditions. Let $\{u_1,u_2\}$ be a basis for S over R, and suppose that the products u_1^2, u_1u_2, u_2u_1, and u_2^2 have been defined. If

$$v = au_1 + bu_2, \quad w = cu_1 + du_2 \tag{3.1}$$

are elements of S, then

$$vw = acu_1^2 + adu_1u_2 + bcu_2u_1 + bdu_2^2; \tag{3.2}$$

i.e., all products are determined. Moreover, they are uniquely determined, since the representations (3.1) are unique. If products u_iu_j are assigned arbitrarily, and multiplication is extended to S by (3.2), multiplication will be right and left distributive over addition, as the reader may easily verify, but will not in general be associative.

Suppose the products u_iu_j are given by

$$\begin{cases} u_1^2 = ru_1 + su_2 & u_2u_1 = mu_1 + nu_2 \\ u_1u_2 = pu_1 + qu_2 & u_2^2 = ku_1 + lu_2. \end{cases} \tag{3.3}$$

We seek conditions on p,q,r,s,m,n,k and l for which the multiplication is associative. The following is routinely verified:

THEOREM 3.4. (A) *The multiplication defined by* (3.2) *and* (3.3) *is associative if and only if multiplication of basis elements is associative; i.e., iff* $(u_h(u_iu_j) = (u_hu_i)u_j$ *for* $j,i,h = 1,2$.

(B) *The multiplication is commutative iff* $u_1u_2 = u_2u_1$; *i.e., iff* $p = m$ *and* $q = n$.

Corresponding to the eight choices of h,i,j in (A) we obtain the following eight conditions:

(3.5)

1. (a) $s = 0$, or (b) $p = m$ and $q = n$.
2. (a) $k = 0$, or (b) $p = m$ and $q = n$.
3. $pq = ks$ and $ps + q^2 = ls + qr$.
4. $pr + qm = mr + np$, and (a) $s = 0$, or (b) $p = m$.
5. $pq = ks$ and $p^2 + qk = kr + lp$.
6. $mn = ks$ and $ms + n^2 = nr + ls$.
7. $mq + nl = np + ql$, and (a) $k = 0$, or (b) $n = q$.
8. $mn = ks$ and $m^2 + kn = kr + lm$.

There are clearly several redundancies among these, and other simplifications are possible. The simplification of these conditions, and breakdown into cases, will be treated in Section 5.

4. The eight isomorphism classes. In this section we shall give abstract characterizations of each of the eight isomorphism classes in terms of the existence of a basis with certain properties, as well as some concrete cases, for use in identifying the rings as they occur in the next section. Sample multiplication formulas are given, taking $u_1 = (1,0)$ and $u_2 = (0,1)$. All multiplications in the class are obtained by varying the the choice of basis vectors. The reader may readily verify, on the basis of the abstract characterizations, that the eight classes are actually distinct isomorphism classes. We list the six commutative cases first, and among these the three with identity first.

It might be noted here that the two-dimensional algebraic extensions $R[x]/(ax^2 + bx + c)$ are rings with R^2 as additive group, and all have identity. These give the three isomorphism classes with identity, according as $ax^2 + bx + c$ is prime, the product of two distinct factors, or is a square in $R[x]$. Cf. [**2**].

CLASS 1. The complex numbers. This, as is well known, is the only field. $R[x]/(ax^2 + bx + c)$ is in this class when $ax^2 + bx + c$ is prime over R.

Abstract characterization: S has a basis u_1, u_2, where u_1 is an identity and $u_2^2 = -u_1$.

Sample multiplication formula in R^2: $(a,b)\,(c,d) = (ac - bd, ad + bc)$.

Matrix representation:

$$u_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and S consists of all real linear combinations of u_1 and u_2.

CLASS 2. The direct sum, $R \oplus R$

Abstract characterization: S has a basis u_1, u_2 of idempotents such that $e = u_1 + u_2$ is an identity, and $u_1u_2 = u_2u_1 = 0$. (u_1 and u_2 generate mutually orthogonal minimal ideals whose direct sum is S.)

$R[x]/(ax^2+bx+c)$ is in this class when $\Delta = b^2-4ac>0$. In this form, $u_1 = (\frac{1}{2} - b/2\Delta^{\frac{1}{2}}) + (a/\Delta^{\frac{1}{2}})x$, $u_2 = 1-u_1$.

Sample multiplication in R^2: $(a,b)(c,d) = (ac,bd)$.

Matrix representation: $u_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $u_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

CLASS 3. $R[x]/(ax^2+bx+c)$ where ax^2+bx+c is a square in $R[x]$. This may be characterized by the fact that there is a basis u_1, u_2 where u_1 is an identity and $u_2^2 = 0$. The only nontrivial ideal is Ru_2.

Sample multiplication formula: $(a,b)\ (c,d) = (ac, ad+bc)$.

Matrix representation: $u_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $u_2 = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$.

CLASS 4. $R\oplus R_0$, where R_0 is R with trivial multiplication. This ring is characterized by the existence of a single nonzero idempotent u_1, and a nonzero annihilator u_2.

Sample multiplication formula: $(a,b)(c,d) = (ac,0)$.

There is no representation in 2×2 matrices, but one in 3×3 matrices is

$$u_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad u_2 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

CLASS 5. This ring is generated by two nilpotents, u_1, u_2, with $u_1^2 = u_2$, $u_1u_2 = u_2u_1 = u_2^2 = 0$. Thus u_2 annihilates S (left and right), and all products of three factors are 0.

Sample multiplication formula: $(a,b)\ (c,d) = (bd,0)$.

Again the simplest matrix representation is in 3×3 matrices:

$$u_1 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix},\quad u_2 = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

CLASS 6. The trivial ring: $(a,b)(c,d) = (0,0)$.

The two noncommutative cases:

CLASS 7. S has a left identity u_1 and a left annihilator u_2. In this ring every element $u_1 + bu_2$ is a left identity, and every element bu_2 is a left annihilator.

Sample multiplication formula: $(a,b)(c,d) = (ac,ad)$.

Matrix representation: $u_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $u_2 = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$.

CLASS 8. This ring is the left-right mirror image of number 7, and is anti-isomorphic to it. There is a right identity u_1 and right annihilator u_2.

Multiplication formula: $(a,b)(c,d) = (ac,bc)$.

Matrix representation: $u_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $u_2 = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$.

5. Reduction to eight classes. The sample multiplication formula given with each isomorphism class in the preceding section shows that there is a multiplication in R^2 for which the resulting ring is in this class. We now wish to show that every ring on R^2 (with continuous multiplication) falls into one of these classes. We do so by simplifying the associativity conditions (3.5), then breaking down into cases, and assigning each case to one of the isomorphism classes.

Note that conditions 1b and 2b are the commutativity conditions. We begin by considering the noncommutative cases. The conditions then become (now renumbered)

1. $s = k = 0$.
2. (a) $p = 0$, or (b) $p \neq 0$ and $q = 0$.
3. (a) $m = 0$, or (b) $m \neq 0$ and $n = 0$.
4. $q^2 = qr$.   5. $n^2 = nr$.
6. $p^2 = pl$.   7. $m^2 = ml$.
8. $pr + qm = mr + np$.   9. $mq + nl = np + ql$.

Conditions 2 and 3 lead to four subcases, which we designate aa, ab, ba, bb.

(aa) $s = k = p = m = 0$. Then $n \neq q$ for noncommutativity.

The remaining conditions reduce to: 4. (a) $q = 0$, or (b) $q = r$; 5. (a) $n = 0$, or (b) $n = \mathrm{r}$; 9. $l = 0$.

4a and 5a will not hold simultaneously since $n \neq q$. The same goes for 4b and 5b. So we have two subcases:

(aaab) $s = k = p = m = q = l = 0$ and $n = r \neq 0$.

(aaba) $s = k = p = m = n = l = 0$ and $q = r \neq 0$.

(ab) $s = k = p = n = 0$, $\mathrm{m} \neq 0$. The remaining conditions reduce to: 7. $m = l \neq 0$; 8. $q = r$. Since $p \neq m$, q may be 0.

(ba) $s = k = q = m = 0$, $p \neq 0$, and 6. $p = l \neq 0$; 8. $n = r$.

(bb) $s = k = q = n = 0$, $p \neq 0$, and $m \neq 0$. For noncommutativity we also need $p \neq m$. But in view of 6. $p = l$ and 7. $m = l$, this case is empty.

In case (aaab) the multiplication equations (3.3) become $u_1^2 = ru_1$, $u_1u_2 = 0$, $u_2u_1 = ru_2$, $u_2^2 = 0$ $(r \neq 0)$. Here $u_1' = r^{-1}u_1$ is a right identity and u_2 a right annihilator, so we are in class 8. The case (aaba) is similarly found to fall into class 7.

In case (ab) we have $u_1^2 = ru_1$, $u_1u_2 = ru_2$, $u_2u_1 = lu_1$, $u_2^2 = lu_2$ $(l \neq 0)$. Here $u_1' = lu_1 - ru_2$ is a left annihilator and $u_2' = l^{-1}u_2$ is a left identity, so class 7. The case (ba) is similarly seen to lie in class 8.

In the commutative case the equations (3.5) reduce to (again renumbered):

(5.1)
1. $p = m$ and $q = n$.   2. $pq = ks$.
3. $ps + q^2 = ls + qr$.   4. $p^2 + qk = kr + pl$.

If $D_1 = qr - ps \neq 0$, one verifies directly that $D_1^{-1}(qu_1 - su_2)$ is an identity. Choose a basis u_1', u_2', where u_1' is the identity. Then there is a homomorphism $R[x] \to S$ determined by $1 \to u_1'$ and $x \to u_2'$. Since $(u_2')^2$ is a linear combination of u_1' and u_2', the kernel is generated by a quadratic polynomial, so S is in class, 1, 2 or 3. If $D_2 = kq - pl \neq 0$, then $D_2^{-1}(ku_1 - pu_2)$ is an identity, and S is similarly found to lie in one of the first three classes. If $D_1 = D_2 = 0$ but $D_3 = rl - sk \neq 0$, then

$p = q = 0$, as one sees by inspection of the column space of the matrix $\left[\begin{smallmatrix} r & p & k \\ s & q & l \end{smallmatrix}\right]$. From (5.1) we have, then, that $ks = kr = ls = 0$. Now $D_3 = rl \neq 0$, so $k = s = 0$. Hence $u_1^2 = ru_1$, $u_2^2 = lu_2$, $u_1u_2 = u_2u_1 = 0$, so $S \cong R \oplus R$, class 2.

Suppose now that $D_1 = D_2 = D_3 = 0$. In this case u_1^2, u_1u_2 and u_2^2 lie in a subspace T of dimension ≤ 1, and hence all products lie in T. If dim $T = 0$ we have class 6. Assume, then, that dim $T = 1$. If a basis u_1', u_2' is chosen so that u_2' is in T, the multiplication equations are $(u_1')^2 = s'u_2'$, $u_1'u_2' = q'u_2'$, $(u_2')^2 = l'u_2$, and the associativity conditions reduce to $(q')^2 = l's'$. There are three possibilities:

(i) $q' = l' = 0$, $s' \neq 0$; (ii) $q' = s' = 0$, $l' \neq 0$; (iii) $q' \neq 0$.

The first is immediately identifiable as class 5. In the second, u_1' is an annihilator and $(l')^{-1}u_2'$ an idempotent, so this is class 4. In the third one may check that $v_1 = -q'u_1' + s'u_2'$ is an annihilator and $(l')^{-1}u_2'$ is still idempotent, so again we have class 4.

6. Rings other than the real numbers. It will occur to some readers to ask for what other rings R the above analysis is valid. The question breaks naturally into two parts: (1) For what other rings R are there eight distinct isomorphism classes corresponding to those in Section 4, and: (2) For what rings R are these the only isomorphism classes? Some results in this direction are listed here; the interested reader should have little difficulty in establishing them for himself. For this section we identify the first three classes by the following representatives:

(6.1) 1. $R[x]/(x^2 + bx + c)$, where $x^2 + bx + c$ is prime in $R[x]$. If R is a subdomain of the reals, take $b^2 - 4c < 0$.
2. $R[x]/(x^2 + x)$.
3. $R[x]/(x^2)$.

THEOREM 6.2. *If R is any subdomain of the real numbers or any of the rings $Z/(k)$, (Z the ring of integers), then the eight classes of Section* 4, *with classes* $1 - 3$ *interpreted as in* (6.1), *are distinct in the sense that no member of one class is isomorphic to a member of a different class.*

THEOREM 6.3. *All rings on the additive group R^2 fall into the eight classes of Section* 4, *with the first three interpreted as in* (6.1), *in each of the following cases*: (A) *R is a subfield of the reals and multiplication in R^2 is continuous*; (B) $R = Z/(p)$, *p a prime*; (C) *R any field, and multiplication in R^2 is subject to the condition* $(ra, rb)(sc, sd) = rs(a, b)(c, d)$.

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THE AFFINE THEOREMS OF PASCH, MENELAUS AND CEVA

FRANCINE ABELES, Newark State College

Introduction. In examining the nature of a geometry, it is particularly important to determine how lines behave when they intersect the sides of a triangle. Moritz Pasch, in the 1880's, was the first to undertake systematically this study. We are specifically interested in the property known as Pasch's theorem.

Consider in a plane p, a triangle ABC and a line l. If l intersects one of the sides of the triangle and does not pass through any of its vertices, then l also intersects at least one of the remaining sides of the triangle.

Since the theorem cannot be proved from Euclid's axioms, it is commonly referred to as an axiom in the literature. We prove the theorem in several ways, including a very simple proof based on Menelaus' theorem.

Pasch's theorem is essential in adequately defining a convex plane. We extend this development to 3-space by generalizing Pasch's theorem, thereby treating convexity as an extension of betweenness. Finally, we state and prove a collinearity theorem equivalent to Menelaus' theorem. The converse of the new theorem is equivalent to Ceva's theorem.

Preliminaries. We need the following information. If A, B, C are distinct collinear points, then $C = aA + bB$, $a + b = 1$, $a,b \neq 0$. A,B,C determine certain invariant ratios, e.g., $AC/CB = b/a$, or $CB/BA = -a/1$. For triangle ABC and point D not on any side, if $D = aA + bB + (1 - a - b)\,C, a, b > 0$, we say that D is a **convex combination** of the frame A,B,C.

PASCH'S THEOREM (Peano's Formulation). *If ABC is a triangle such that $B - C - F$, i.e., C is between points B and F, and $C - E - A$, then there is a point D on line FE such that $A - D - B$.*

Proof. We will show that a line intersecting side AC internally and BC externally must intersect AB internally.

(1) $F \in BC \Rightarrow F = eB + (1-e)C$, or $BF/FC = (1-e)/e$.
Since $B - C - F$, the ratio is negative.

(2) $E \in CA \Rightarrow E = cC + (1-c)A$, or $CE/EA = (1-c)/c$.
Since $C - E - A$, the ratio is positive.

(3) Let $P \in AB$ and $A - P - B$ then $P = aA + (1-a)B$, $AP/PB = (1-a)/a$, and the ratio is positive.

(4) Let $D \in$ line FE and $D - E - F$ then $D = tE + (1-t)F$, $FD/DE = t/(1-t)$ and the ratio is negative.

Substituting (1) and (2) in (4), we obtain

$$D = t(1-c)A + e(1-t)B + [tc + (1-t)(1-e)]C.$$

If $D = P$, we must have

(5) $t(1-c) = a$, or $tc = t - a$.

(6) $e(1-t) = 1-a$, or $te = e + a - 1$.

(7) $tc + (1-t)(1-e) = 0$.

Substituting (5) and (6) in (7), we see that $t - a + 1 - e - t + e + a - 1 = 0$.

REMARKS. Pasch's theorem can easily be generalized to 3-space. But it makes no sense to do so if our only concern is with the theorem's significance as a plane axiom of order. Note that the order axiom below is just a more formal version of Pasch's Theorem.

Let A, B, C be three noncollinear points in a plane p and let l be a line in p. If l lies between A and B, and C is not on l, then l lies between B and C or between A and C [**1**, p. 42].

However, Pasch's theorem can also be used to prove a plane separation theorem and conversely. *For any line k in a plane p, the set $p-k$ can be uniquely represented as the union of two nonempty, disjoint sets W_1, W_2 satisfying the two conditions*:

(i) *If $P, Q \in W_1$ or $P, Q \in W_2$, then k does not lie between P and Q.*

(ii) *If $P \in W_1$ and $Q \in W_2$, then k lies between P and Q* [**1**, p. 44].

It is in this sense that we now state and prove a 3-space separation theorem.

THEOREM. *If $P_0P_1P_2P_3$ is a tetrahedron, Q_0 is in the plane of $P_1P_2P_3$, Q_1 is in the plane of $P_0P_2P_3$, Q_2 in the plane of $P_0P_1P_3$, no Q_i is on an edge of the tetrahedron, and if S, the plane of $Q_0Q_1Q_2$, intersects plane $P_0P_2P_3$ and faces $P_1P_2P_3$ and $P_0P_1P_3$, then there is a point Q_3 in S such that $Q_3 \in$ face $P_0P_1P_2$.*

Proof. (8) $Q_0 \in \{P_1P_2P_3\} \Rightarrow Q_0 = aP_1 + bP_2 + (1-a-b)P_3$.
Since Q_0 is a convex combination of $P_1P_2P_3$, $a, b > 0$.

(9) $Q_1 \in \{P_0P_2P_3\} \Rightarrow Q_1 = cP_0 + dP_2 + (1-c-d)P_3$.

(10) $Q_2 \in \{P_0P_1P_3\} \Rightarrow Q_2 = eP_0 + fP_1 + (1-e-f)P_3$, $e, f > 0$.

(11) $Q_3 \in S \Rightarrow Q_3 = sQ_0 + tQ_1 + (1-s-t)Q_2$.

Let $R \in \{P_0P_1P_2\}$ such that R is a convex combination of $P_0P_1P_2$.

(12) Then $R = gP_0 + hP_1 + (1-g-h)P_2$, $g, h > 0$.

We will show $Q_3 = R$. Substituting (8), (9), (10) in (11), we obtain

$$\begin{aligned} Q_3 &= s[aP_1 + bP_2 + (1-a-b)P_3] + t[cP_0 + dP_2 + (1-c-d)P_3] \\ &\quad + (1-s-t)[eP_0 + fP_1 + (1-e-f)P_3] \\ &= [tc + e(1-s-t)]P_0 + [sa + f(1-s-t)]P_1 + (sb + td)P_2 \\ &\quad + [s(1-a-b) + t(1-c-d) + (1-s-t)(1-e-f)]P_3. \end{aligned}$$

For $Q_3 = R$, we must have

(13) $tc + e(1-s-t) = g$, or $e(1-s-t) = g - tc$,

(14) $sa + f(1-s-t) = h$, or $f(1-s-t) = h - sa$,

(15) $sb + td = 1 - g - h$,

(16) $s(1-a-b) + t(1-c-d) + (1-s-t)(1-e-f) = 0$, or
$s(1-a) + t(1-c) - (sb + td) - e(1-s-t) - f(1-s-t) + (1-s-t) = 0$.

Substituting (13), (14), (15) in (16), we have

$$s - sa + t - tc - 1 + g + h - g + tc - h + sa + 1 - s - t = 0.$$

Returning to the original formulation of Pasch's theorem, we find that it can be proved in a more analytic fashion. We use equations (1), (2) and (3). If $P = D$, then P, E, F are collinear $\Rightarrow$ area of triangle PFE, denoted $[PFE]$, $= 0$. We multiply the three equations together, omitting terms like $[AAB]$, since it is obvious that AAB is a degenerate triangle whose area is zero. Permutations of the elements of $[ABC]$ in the same order yield the same area, while a permutation involving an odd exchange of elements yields the negative of the area, e.g., $[CBA] = -[ABC]$.

Proof. $[PFE] = (aA + (1-a)B)(eB + (1-e)C)$
$(cC + (1-c)A) = (ace + (1-a)(1-e)(1-c))[ABC]$.
Since $[ABC] \neq 0$, $ace = -(1-a)(1-e)(1-c)$.

Determinants can be used to express collinearity and area notions. We can write $[FEP]$ in determinant form as an affine combination of the points of the frame A, B, C. We have

$$[FEP] = \begin{vmatrix} 0 & e & 1-e \\ 1-c & 0 & c \\ a & 1-a & 0 \end{vmatrix}.$$

So P, E, F collinear if and only if $[FEP] = 0$. This method can also be used to prove Pasch's theorem in 3-space, i.e., if $Q_3 = R$, then Q_0, Q_1, Q_2, R will be coplanar $\Leftrightarrow$ volume of $Q_0Q_1Q_2R = 0$, or using equations (8), (9), (10), (12),

$$[Q_0Q_1Q_2R] = \begin{vmatrix} 0 & a & b & 1-a-b \\ c & 0 & d & 1-c-d \\ e & f & 0 & 1-e-f \\ g & h & 1-g-h & 0 \end{vmatrix} = 0.$$

Our final proof of Pasch's theorem uses the classical theorem of Menelaus.

If r_1, r_2, r_3 are the ratios in which points P, E, F divide sides CB, BA, AC of a triangle, respectively, the points are collinear if and only if $r_1r_2r_3 = -1$.

Proof. We use the ratios (1), (2), (3). P, E, F collinear if and only if $(1-e)(1-c)(1-a)/eca = -1$, i.e., $-(1-a)(1-e)(1-c) = ace$.

Menelaus' collinearity theorem has been studied extensively and several generalizations have been proved, e.g., Carnot's theorem [**2**, p. 309]. A similar statement can be made for Ceva's concurrence theorem. It is possible to prove a theorem and its converse which are equivalent to Menelaus' theorem and Ceva's theorem, respectively [**3**, p. 798].

THEOREM. *Let ABC be a triangle and let A', B', C', and Q be distinct points in the plane of ABC. If $QA'/A'A + QB'/B'B + QC'/C'C = -1$, then A', B', C' are on the rays BC, CA, AB, and collinear with A,Q; B,Q; C,Q, respectively, and conversely.*

Proof. Let

$$QA'/A'A = -a, \text{ or } Q = aA + (1-a)A', \tag{17}$$

$$QB'/B'B = -b, \text{ or } Q = bB + (1-b)B', \tag{18}$$

$$QC'/C'C = -c, \text{ or } Q = cC + (1-c)C' \tag{19}$$

where $-a - b - c = -1$. Q,A,A'; Q,B,B'; Q,C,C' are collinear.

Let

$$S_1 \in \text{ray } BC, \text{ so } S_1 = s_1B + (1-s_1)C,$$

$$S_2 \in \text{ray } CA, \text{ so } S_2 = s_2C + (1-s_2)A,$$

$$S_3 \in \text{ray } AB, \text{ so } S_3 = s_3A + (1-s_3)B.$$

Q is in the plane of A, B, C, so

$$Q = aA + bB + cC, \qquad a + b + c = 1$$

$$= aA + (1-a)\left[\frac{b}{1-a}B + \frac{c}{1-a}C\right].$$

Setting $S_1 = \dfrac{b}{1-a}B + \dfrac{c}{1-a}C$, we have

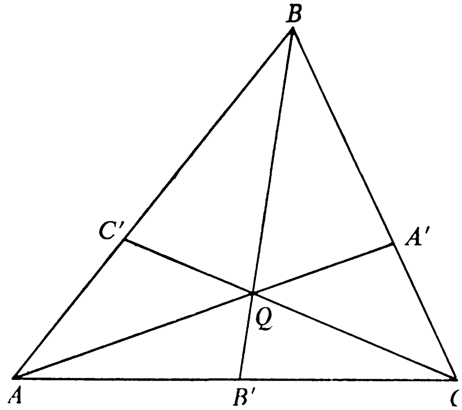
$$Q = aA + (1-a)S_1.$$

Similarly, $Q = bB + (1-b)S_2$ and $Q = cC + (1-c)S_3$.

Comparing these equations with (17), (18) and (19), we see

$$S_1 = A', \; S_2 = B', \; S_3 = C'.$$

We leave the proof of the converse to the reader. Note that C', Q, C are menelaus points for the sides of triangle ABB', for example, while AA', BB', CC' are cevian lines of triangle ABC. Refer to the figure on page 82.



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AN OCCUPANCY PROBLEM INVOLVING PLACEMENT OF PAIRS OF BALLS

ALVIN D. WIGGINS, University of California, Davis

1. Introduction. Suppose that we have $2k$ balls which must be placed in n linearly arranged cells with the following restrictions ($n \geqq 2k$):

(a) Each ball must be placed in some cell.

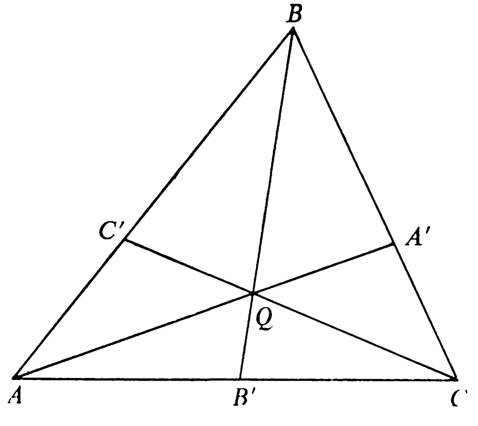
(b) Each cell must contain no more than one ball.

(c) Balls must be placed in pairs, and only in pairs of adjacent unoccupied cells.

The problem is to determine the total number, $N_{n,k}^{(2)}$, of ways in which this may be accomplished.

This problem had its origin in an application to a problem in physical chemistry. Since I am not a physical chemist I shall not presume to explain the complexities of that discipline which led to the present problem. Rather, I shall formulate the problem in another way which will make sense to practically everyone. Suppose that at a concert in the park there is a row of n seats. Then it is well known that the total number of ways of seating $k(\leqq n)$ persons is $\binom{n}{k}$. Suppose however that a row of n seats is reserved for couples only. The problem now is to count the number of ways of seating $k(\leqq n/2)$ couples if no couple is willing to be separated.

Section 2 contains the solution of the occupancy problem in the case in which balls are placed in pairs. Section 3 contains the solution to the problem of deter-



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The problem is to determine the total number, $N_{n,k}^{(2)}$, of ways in which this may be accomplished.

This problem had its origin in an application to a problem in physical chemistry. Since I am not a physical chemist I shall not presume to explain the complexities of that discipline which led to the present problem. Rather, I shall formulate the problem in another way which will make sense to practically everyone. Suppose that at a concert in the park there is a row of n seats. Then it is well known that the total number of ways of seating $k(\leqq n)$ persons is $\binom{n}{k}$. Suppose however that a row of n seats is reserved for couples only. The problem now is to count the number of ways of seating $k(\leqq n/2)$ couples if no couple is willing to be separated.

Section 2 contains the solution of the occupancy problem in the case in which balls are placed in pairs. Section 3 contains the solution to the problem of determining the average number of remaining pairs of adjacent seats, under a suitable definition of "average", after the seat row has partially filled up. Section 4 consists of a brief observation on the generalization to the situation in which a row of n seats is reserved for parties of exactly $m(\leqq n)$ persons.

2. Solution in the case of pairs of balls. Let $N_{n,k}^{(2)}$ denote the number of ways in which $2k$ balls can be placed in n cells subject to the restrictions enumerated in the preceding section. A little reflection reveals that the problem posed is equivalent to the problem of counting the total number of dyadic integers consisting of $n+1$ ones and $2k$ zeros, in which the n spaces between the $n+1$ ones play the role of cells, and the $2k$ zeros are analogous to the balls, and the restrictions above are modified in an obvious way. As an example, for $n = 14$ and $k = 3$, one such dyadic number will be written

$$11010111010111101011. \tag{2.1}$$

Examples of sequences of ones and zeros which are *not* eligible to be counted are

$$10011, \tag{2.2}$$

$$1011011. \tag{2.3}$$

Since we are concerned with the total number of *distinct* ways in which the placement of the zeros can be accomplished we can visualize beginning by placing the first pair of zeros in the first two spaces on the extreme left hand end of the sequence of uninterrupted ones, and thereafter placing successive pairs of zeros always to the right of the last pair placed. In this way every admissible sequence is counted exactly once. This method of counting can serve to motivate the writing of a recursion formula. Specifically, we have

$$N_{n,k}^{(2)} = N_{n-2,k-1}^{(2)} + N_{n-3,k-1}^{(2)} + \cdots + N_{2k-2,k-1}^{(2)}. \tag{2.4}$$

For $k = 1$ and any $n \geqq 2$, we have, obviously

$$N_{n,1}^{(2)} = n-1. \tag{2.5}$$

For $k = 2$, we apply (2.4) and (2.5) to find for all $n \geqq 4$

$$\begin{aligned} N_{n,2}^{(2)} &= \sum_{i=2}^{n-2} N_{n-i,1}^{(2)} = \sum_{i=2}^{n-2} (n-i-1) = \sum_{j=1}^{n-3} j = \\ &= (n-3)(n-2)/2 = \binom{n-2}{2}. \end{aligned} \tag{2.6}$$

In view of (2.6) and the fact that (2.5) can be written $N_{n,1}^{(2)} = n-1 = \binom{n-1}{1}$, this leads to the conjecture

$$N_{n,k}^{(2)} = \binom{n-k}{k}, \quad 1 \leqq k \leqq n/2. \tag{2.7}$$

That this conjecture is correct can easily be shown by induction on k. Accordingly we take (2.7) as the induction hypothesis. Then for $k+1$ and all $n \geqq 2k+2$, we have, applying (2.4) and (2.7)

$$N_{n,k+1}^{(2)} = \sum_{i=2}^{n-2k} N_{n-i,k}^{(2)} = \sum_{i=2}^{n-2k} \binom{n-i-k}{k} = \sum_{j=k}^{n-k-2} \binom{j}{k}$$

$$= \binom{n-k-1}{k+1} = \binom{n-(k+1)}{k+1},$$

and the proof is complete.

3. Calculation of the average number of additional cell pairs available for the placement of one more pair of balls. For a fixed integer couple, (n,k), we have seen that $N_{n,k}^{(2)} = \binom{n-k}{k}$ is the total number of ways of placing k pairs of balls in n cells subject to the restrictions in 1(a), 1(b), 1(c). Equivalently this is the total number of ways of writing a restricted set of dyadic integers, or sequences of zeros and ones, using $2k$ zeros and $n+1$ ones. Assume now that the totality of such sequences is displayed in some manner, and that each sequence of the set is uniquely identified by an integer i where $i = 1,2,3,\cdots,N_{n,k}^{(2)}$. For a fixed integer i, associate with the ith sequence a nonnegative integer $x_{n,k,i}^{(2)}$ which represents the number of ways in which an additional pair of zeros can be placed in the sequence, subject always to the restrictions mentioned above. Now define the integer $M_{n,k}^{(2)}$ to be the sum of all the $x_{n,k,i}^{(2)}$:

$$M_{n,k}^{(2)} = \sum_{i=1}^{N_{n,k}^{(2)}} x_{n,k,i}^{(2)}. \tag{3.1}$$

We can then define the average number of additional available cell pairs as

$$A_{n,k}^{(2)} = M_{n,k}^{(2)}/N_{n,k}^{(2)}. \tag{3.2}$$

Thus the main problem is the determination of $M_{n,k}^{(2)}$.

For n and k fixed denote by $\mathscr{T}_{n,k}^{(2)}$ the set of all sequences of zeros and ones which are admissible according to the restrictions (1a), (1b), and (1c). Consider the subclass $\mathscr{S}_{n,k}^{(2)}$ of all sequences of $\mathscr{T}_{n,k}^{(2)}$ in which it is possible to insert an additional pair of adjacent zeros. The cardinality of this set is $M_{n,k}^{(2)}$. Obviously each sequence of $\mathscr{S}_{n,k}^{(2)}$ will yield a sequence of $\mathscr{T}_{n,k+1}^{(2)}$. Conversely, consider the sequences of $\mathscr{T}_{n,k+1}^{(2)}$, and number the $k+1$ pairs of adjacent zeros, say, from left to right. The *removal* of exactly one pair of adjacent zeros from any sequence of $\mathscr{T}_{n,k+1}^{(2)}$ will yield a sequence of $\mathscr{S}_{n,k}^{(2)}$. Thus the mapping of $\mathscr{S}_{n,k}^{(2)}$ to $\mathscr{T}_{n,k+1}^{(2)}$ is one-to-one and onto, consequently $\mathscr{S}_{n,k}^{(2)}$ and $\mathscr{T}_{n,k+1}^{(2)}$ have the same cardinality. Since $N_{n,k+1}^{(2)} = \binom{n-k-1}{k+1}$, and the removal of a pair of adjacent zeros from an arbitrary sequence of $\mathscr{T}_{n,k+1}^{(2)}$ can be accomplished in $k+1$ ways, we have

$$M_{n,k}^{(2)} = (k+1)N_{n,k+1}^{(2)} = (k+1)\binom{n-k-1}{k+1}.$$

Finally, substituting in (3.2) we have for the average number of additional cell pairs

(or couples who can be seated at the concert)

$$A_{n,k}^{(2)} = (n-2k)(n-2k-1)/(n-k).$$

4. Generalization to the case of insertion of *m* zeros at a time. The generalization to the case in which not pairs of zeros, but m zeros are inserted simultaneously into the sequence of ones is straightforward. It consists of replacing the word or concept of "pair" and its mathematical cognates by the word or concept of "m-tuple" and its mathematical cognates in the foregoing development. The results are

$$N_{n,k}^{(m)} = \binom{n-(m-1)k}{k}, \tag{4.1}$$

$$M_{n,k}^{m} = (k+1)N_{n,k+1}^{(m)} = (k+1)\binom{n-(m-1)(k+1)}{k+1}, \tag{4.2}$$

$$A_{n,k}^{(m)} = \frac{(n-(m-1)(k+1))!\,(n-mk)!}{(n-m(k+1))!\,(n-(m-1)k)!} \tag{4.3}$$

$$= (k+1)\binom{n-(m-1)(k+1)}{k+1} \Big/ \binom{n-(m-1)k}{k}.$$

Acknowledgment. I am indebted to the referee for making a number of suggestions for shortening and clarifying the manuscript.

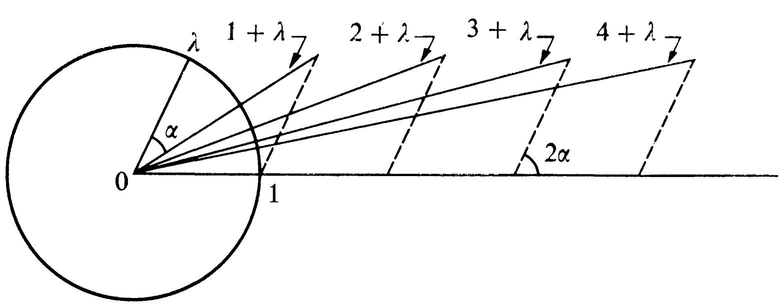
Reference

1. W. Feller, An Introduction to Probability Theory and its Applications, vol. I, 2nd. ed., Wiley, New York, 1957.

THE ASYMPTOTIC BEHAVIOR OF A CERTAIN PRODUCT

MING-CHIT LIU, University of Hong Kong

For $\alpha \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$, let $\lambda = e^{i2\alpha}$. It is interesting to ask what is the asymptotic behavior of the product $\prod_{k=1}^{n-1} |k+\lambda|$. When $\alpha = 0$ the result is already known.



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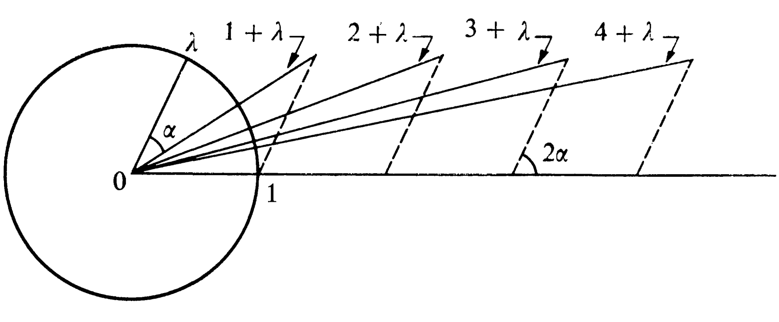

That is Stirling's formula

$$\prod_{k=1}^{n-1} |k+1| = n! = O(n^{n+\frac{1}{2}}e^{-n}),$$

as $n \to \infty$.

On the other hand, if we consider $(\prod_{k=1}^{n-1}|k+\lambda|)/(n-1)!$ we see that

$$\frac{\prod_{k=1}^{n-1} |k+\lambda|}{(n-1)!} \leq \frac{\prod_{k=1}^{n-1} (k+|\lambda|)}{(n-1)!} = n$$

for any $\alpha \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$. The above inequalities reveal that if $\alpha \neq 0$ then $(\prod_{k=1}^{n-1}|k+\lambda|)/(n-1)!$ may tend to infinity not so fast as n does and for some α it may even be bounded. This means that there may be some real-valued function $T(\alpha)$ such that $|T(\alpha)| \leq 1$ and

$$\frac{\prod_{k=1}^{n-1} |k+\lambda|}{(n-1)!} = O(n^{T(\alpha)})$$

as $n \to \infty$. So if we know the asymptotic behavior of $\prod_{k=1}^{n-1}|k+\lambda|$ then immediately we obtain the function $T(\alpha)$. In this note we shall prove:

THEOREM. *For* $\alpha \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$, *let* $\lambda = e^{i2\alpha}$. *Then we have*

$$\prod_{k=1}^{n-1} |k+\lambda| = O(n^{n-\frac{1}{2}+\cos 2\alpha}\, e^{-n})$$

as $n \to \infty$.

As a consequence of our theorem we have

$$\frac{\prod_{k=1}^{n-1} |k+\lambda|}{(n-1)!} = O\left(\frac{n^{n-\frac{1}{2}+\cos 2\alpha}e^{-n}}{n^{n-\frac{1}{2}}e^{-n}}\right) = O(n^{\cos 2\alpha})$$

and $T(\alpha) = \cos 2\alpha$.

We need a lemma.

LEMMA. *Let A be any real number and n positive integer. Then*

$$D_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \log(x^2+2Ax+1)\,dx - \log(n^2+2An+1)$$
$$= O(n^{-2})$$

as $n \to \infty$.

When one considers the area under the curve $\log(x^2+2Ax+1)$ between $n-\frac{1}{2}$ and $n+\frac{1}{2}$ and a corresponding rectangle of height $\log(n^2+2An+1)$, one sees geometrically that this difference becomes small as n becomes large. L'Hospital's rule will establish this formally. Furthermore, by applying the law of the mean for integrals one quickly sees that the difference goes to zero at least as fast as $1/n$, and with a bit more observation that $n \cdot D_n$ goes to zero as well.

Proof of the Lemma. Write

$$D_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\,dx,$$

where

$$f(x) = \log\left(\frac{x^2+2Ax+1}{n^2+2An+1}\right).$$

We see that

$$f(n) = 0, \qquad f'(x) = \frac{2(x+A)}{x^2+2Ax+1},$$

$$f''(n) = \frac{2(-n^2-2An+1-2A^2)}{(n^2+2An+1)^2} = O(n^{-2}),$$

$$f'''(n) = \frac{4(n^3+3An^2+3(2A^2-1)n+A(4A^2-3))}{(n^2+2An+1)^3} = O(n^{-3}),$$

as $n \to \infty$. It follows from Taylor's formula with remainder that

$$D_n = f'(n)\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}(x-n)\,dx + \tfrac{1}{2}f''(n)\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}(x-n)^2dx$$
$$+\tfrac{1}{6}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f'''(c)(x-n)^3\,dx,$$

where c is between x and n. Since

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |f'''(c)|\,|x-n|^3dx \leqq \max_{c\in[n-\frac{1}{2},n+\frac{1}{2}]} |f'''(c)| = O(n^{-3})$$

and

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}(x-n)\,dx = 0$$

we have that $D_n = O(n^{-2})$ as $n \to \infty$. This proves the lemma.

We come now to the proof of the theorem. We have

$$\log\prod_{k=1}^{n-1}|k+\lambda|^2 = \sum_{k=1}^{n-1}\log(k^2+2k\cos 2\alpha+1)$$
$$= \int_{\frac{1}{2}}^{n-\frac{1}{2}}\log(x^2+2x\cos 2\alpha+1)\,dx$$
$$+\sum_{k=1}^{n-1}\log(k^2+2k\cos 2\alpha+1) - \sum_{k=1}^{n-1}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}\log(x^2+2x\cos 2\alpha+1)\,dx$$

$$= J_n - \sum_{k=1}^{n-1} D_k,$$

say, where

$$J_n = \int_{\frac{1}{2}}^{n-\frac{1}{2}} \log(x^2 + 2x\cos 2\alpha + 1)\,dx$$
$$= (n - \tfrac{1}{2} + \cos 2\alpha)\log\{(n-\tfrac{1}{2})^2 + 2(n-\tfrac{1}{2})\cos 2\alpha + 1\} - 2n + O(1)$$
$$= (2n - 1 + 2\cos 2\alpha)\log n - 2n + O(1).$$

By our lemma we have $\sum_{k=1}^{n-1} D_k = O(1)$. Hence,

$$\log \prod_{k=1}^{n-1} |k + \lambda|^2 = (2n - 1 + 2\cos 2\alpha)\log n - 2n + O(1).$$

This proves the theorem.

REMARK. The asymptotic behavior of $(\prod_{k=1}^{n-1} |k+\lambda|)/(n-1)!$ is interesting. The motivation is as follows:

L. Špaček introduced the class of spirallike functions, S_α. For $|z| < 1$ we say $f(z) \in S_\alpha$ if (i) $f(z)$ is analytic, (ii) $f(0) = 0$ and $f'(0) = 1$, i.e., $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, (iii) $Re\{e^{i\alpha} z f'(z)/f(z)\} \geqq 0$.

It has been proved [1] that if $f(z) \in S_\alpha$ then $f(z)$ is schlicht (i.e., $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$) and the Bieberbach conjecture, $|a_n| \leqq n$ is satisfied. The sharp estimate on $|a_n|$ was obtained by J. Zamorski [2]. The result is

$$|a_n| \leqq \prod_{k=0}^{n-2} \frac{|k + 2\cos\alpha\, e^{i\alpha}|}{k+1}.$$

But $1 + \lambda = |1 + \lambda| e^{i\alpha} = 2\cos\alpha\, e^{i\alpha}$. We have for any $\alpha \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$

$$|a_n| \leqq \frac{\prod_{k=1}^{n-1} |k+\lambda|}{(n-1)!}.$$

From this we can see that the Bieberbach conjecture is satisfied for the class of spirallike functions, i.e., $|a_n| \leqq n$ for any $\alpha \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$. But it is natural to expect that for different values of α the bounds of $|a_n|$ should be different. Really, it follows from our theorem that $|a_n| = O(n^{\cos 2\alpha})$. Except when $\alpha = 0$ the coefficients a_n grow at a rate smaller than n and for a certain range of α, a_n are even bounded.

My thanks are due to the referee for his valuable suggestions.

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2. J. Zamorski, About the extremal spiral schlicht functions, Ann. Polon. Math., 9 (1961) 265–273.

ELEMENTARY INEQUALITIES FOR INTEGRALS

C. J. ELIEZER, La Trobe University, Australia

1. Inequalities of interest may be obtained by integrating, under appropriate conditions, known inequalities for functions. The object of this note is to consider explicitly two such inequalities which seem to be both simple and useful, and which are suggestive of further generalizations.

Let $f(x)$ be a nonnegative function of a real variable x, and let S be a set of real numbers such that the integral

$$I_k \equiv \int f^k \equiv \int_a^b \{f(x)\}^k dx,$$

taken over an appropriate range (a, b), with $b > a$, exists when $k \in S$. Then

$$\int f^m + \int f^n \geqq 2 \int f^{(m+n)/2} \tag{1}$$

$$\int f^m \int f^n \geqq \left[\int f^{(m+n)/2}\right]^2, \tag{2}$$

where m, n and $\frac{1}{2}(m+n)$ belong to the set S.

Inequality (1) may be proved by starting with the Arithmetic Mean – Geometric Mean inequality, and (2) is a particular case of Schwarz inequality. Both of these may also be independently derived by using properties of convex functions. [References **1** and **2**.]

2. Two particular cases are worthy of note.

(i) Writing $\delta = b - a =$ range of integration, and taking $m = 1$, $n = -1$, we have

$$\int f + \int \frac{1}{f} \geqq 2\delta \tag{3}$$

$$\left(\int f\right)\left(\int \frac{1}{f}\right) \geqq \delta^2. \tag{4}$$

(ii) When $m = 2$, $n = 0$, (2) gives

$$\int f^2 \int 1 \geqq \left(\int f\right)^2. \tag{5}$$

A geometrical representation of this may be obtained as follows: Let PQ be the arc of the curve $y = f(x)$ between $x = a$ and $x = b$. Let A be the area bounded by the arc PQ, the ordinates at P and Q and the x-axis; and let V be the volume of the solid of revolution formed by rotating the area A about the x-axis. Then inequality (5) is equivalent to

$$V\delta \geqq \pi A^2. \tag{6}$$

3. Some interesting properties of the integrals I_k may be deduced from these inequalities. For example, suppose we wish to consider the behavior of I_n as $n \to \infty$.

From (1), with $m = n + 2$, we have

$$I_{n+2} + I_n \geqq 2I_{n+1} \tag{7}$$

which may be written

$$I_{n+2} - I_{n+1} \geqq I_{n+1} - I_n. \tag{8}$$

Thus $I_{n+1} - I_n$ is monotonic increasing with n, and hence would tend either to a finite limit L or to $+\infty$ as $n \to \infty$.

If this limit is infinity, then I_n is ultimately increasing and tends to infinity as $n \to \infty$. If the limit L is finite and $L > 0$, then we deduce $I_n \to \infty$ as $n \to \infty$, and $I_n \geqq I_0 + nL$. If $L \leqq 0$, then we can show that $L = 0$, I_n ultimately decreases as n increases, and $I_n \to l \geqq 0$ as $n \to \infty$. The same results may be obtained from inequality (2), which gives

$$\frac{I_{n+2}}{I_{n+1}} \geqq \frac{I_{n+1}}{I_n}, \tag{9}$$

showing that $(I_{n+1})/I_n$ is monotonic increasing.

4. By taking different forms of the function f, we should be able to obtain inequalities of interest.

For example:

(i) $f(t) = e^t$, $a = -\infty$, $b = 0$, m, n positive numbers. From (1)

$$\frac{1}{m} + \frac{1}{n} \geqq \frac{2}{m+n}. \tag{10}$$

(ii) $f(t) = e^t$, $a = -1$, $b = 1$, m, n any real numbers, (1) gives

$$\frac{\sinh m}{m} + \frac{\sinh n}{n} \geqq \frac{2\sinh(m+n)/2}{m+n}. \tag{11}$$

(iii) $f(t) = te^{-t}$, $a = 0$, $b = \infty$. From (2)

$$\frac{\Gamma(m+1)\Gamma(n+1)}{\left\{\Gamma\left(\frac{m+n}{2}+1\right)\right\}^2} \geqq \frac{m^{n+1}n^{n+1}}{\left(\frac{m+n}{2}\right)^{m+n+2}}. \tag{12}$$

(iv) $f(t) = \sqrt{(1 - k^2\sin^2 t)}$, $a = 0$, $b = x$. Then (3) and (4) give

$$E(x,k) + F(x,k) \geqq 2x \tag{13}$$

$$E(x,k)F(x,k) \geqq x^2, \tag{14}$$

where E and F are elliptic integrals of the first and second kind.

5. As a slightly more general form of (1) and (2) we have

$$\int gf^{m} + \int gf^{n} \geqq 2 \int gf^{(m+n)/2} \tag{15}$$

$$\left(\int gf^{m}\right)\left(\int gf^{n}\right) \geqq \left(\int gf^{(m+n)/2}\right)^{2}, \tag{16}$$

where f is as before, and $g(x)$ is a nonnegative function of x. If we take $g(t) = e^{-t}$. $f(t) = t$, $a = 0$, $b = \infty$, (16) gives

$$\Gamma(m+1)\Gamma(n+1) \geqq \left\{\Gamma\left(\frac{m+n}{2}+1\right)\right\}^{2}. \tag{17}$$

This may be compared with (12) above.

A variety of inequalities may be obtained from further generalizations.

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1. C. J. Eliezer and D. E. Daykin, Generalizations and applications of Cauchy-Schwarz inequalities, Quart. J. Math., 18, 72 (1967) 357–360.

2. D. E. Daykin and C. J. Eliezer, Generalizations of the A. M. and G. M. inequality, this MAGAZINE, 40 (1967) 247–250.

A RECURSIVE FORMULA FOR THE NUMBER OF PARTITIONS OF AN INTEGER N INTO m UNEQUAL INTEGRAL PARTS

VÁCLAV KONEČNÝ, Jarvis Christian College, Hawkins

Denote by $Q_k(m,N)$ the number of partitions of the positive integer N into m unequal integral parts, each integer in the partition coming from the interval $[k,N]$ where k is an integer $\geqq 1$. If $\{x_i\}_{i=1}^{m}$ is such a partition, we will assume that $x_i < x_{i+1}$ for $i = 1, 2, \cdots m-1$. We call x_1 the first term of the partition and we denote the set of all such partitions by $S_k(m,N)$.

LEMMA 1. $Q_k(m,N) = Q_1(m, N-(k-1)m)$.

Proof. $\Sigma_{i=1}^{m} x_i = N$ if and only if $\Sigma_{i=1}^{m}(x_i - (k-1)) = N-(k-1)m$. Thus there exists a 1-1 correspondence between the sets $S_k(m,N)$ and $S_1(m, N-(k-1)m)$. We conclude that $Q_k(m,N) = Q_1(m,N-(k-1)m)$.

LEMMA 2. $Q_1(m,N) = \Sigma_q Q_1(m-1, N-qm)$ *where the summation is over all* q *such that* q *is the first term in a partition in* $S_1(m,N)$.

Proof. Let $\{x_i\}_{i=1}^{m}$ be a partition in $S_1(m,N)$ with the first term $x_1 = q$. Then $\Sigma_{i=2}^{m} x_i = N-q$ and thus $\{x_i\}_{i=2}^{m} \in S_{q+1}(m-1, N-q)$. The converse is also true. But $Q_{q+1}(m-1,N-q) = Q_1(m-1,N-qm)$ by Lemma 1. If we sum over all q, we get the asserted result.

$$\int gf^m + \int gf^n \geqq 2 \int gf^{(m+n)/2} \tag{15}$$

$$\left(\int gf^m\right)\left(\int gf^n\right) \geqq \left(\int gf^{(m+n)/2}\right)^2, \tag{16}$$

where f is as before, and $g(x)$ is a nonnegative function of x. If we take $g(t) = e^{-t}$. $f(t) = t$, $a = 0$, $b = \infty$, (16) gives

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A RECURSIVE FORMULA FOR THE NUMBER OF PARTITIONS OF AN INTEGER N INTO m UNEQUAL INTEGRAL PARTS

VÁCLAV KONEČNÝ, Jarvis Christian College, Hawkins

Denote by $Q_k(m, N)$ the number of partitions of the positive integer N into m unequal integral parts, each integer in the partition coming from the interval $[k, N]$ where k is an integer $\geqq 1$. If $\{x_i\}_{i=1}^m$ is such a partition, we will assume that $x_i < x_{i+1}$ for $i = 1, 2, \cdots m-1$. We call x_1 the first term of the partition and we denote the set of all such partitions by $S_k(m, N)$.

LEMMA 1. $Q_k(m, N) = Q_1(m, N-(k-1)m)$.

Proof. $\Sigma_{i=1}^m x_i = N$ if and only if $\Sigma_{i=1}^m (x_i - (k-1)) = N - (k-1)m$. Thus there exists a 1-1 correspondence between the sets $S_k(m, N)$ and $S_1(m, N-(k-1)m)$. We conclude that $Q_k(m, N) = Q_1(m, N-(k-1)m)$.

LEMMA 2. $Q_1(m, N) = \Sigma_q Q_1(m-1, N-qm)$ *where the summation is over all* q *such that* q *is the first term in a partition in* $S_1(m, N)$.

Proof. Let $\{x_i\}_{i=1}^m$ be a partition in $S_1(m, N)$ with the first term $x_1 = q$. Then $\Sigma_{i=2}^m x_i = N - q$ and thus $\{x_i\}_{i=2}^m \in S_{q+1}(m-1, N-q)$. The converse is also true. But $Q_{q+1}(m-1, N-q) = Q_1(m-1, N-qm)$ by Lemma 1. If we sum over all q, we get the asserted result.

Number of Partitions of N into m parts

N \ m	1	2	3	4	5	6	7	8	9	10	11	12	13
76	1	37	444	2484	7599	13702	14950	7949	3589	653	42	0	0
77	1	38	456	2586	8056	14800	16475	11018	4206	807	56	0	0
78	1	38	469	2700	8529	15944	18138	12450	4904	984	76	1	0
79	1	39	481	2808	9027	17180	19928	14012	5708	1204	99	1	0
80	1	39	494	2928	9542	18467	21873	15765	6615	1455	131	2	0
81	1	40	507	3042	10083	19858	23961	17674	7657	1761	169	3	0
82	1	40	520	3169	10642	21301	26226	19805	8824	2112	219	5	0
83	1	41	533	3289	11229	22856	28652	22122	10156	2534	278	7	0
84	1	41	547	3422	11835	24473	31275	24699	11648	3015	355	11	0
85	1	42	560	3549	12470	26207	34082	27493	13338	3590	445	15	0
86	1	42	574	3689	13125	28009	37108	30588	15224	4242	560	22	0
87	1	43	588	3822	13811	29941	40340	33940	17354	5013	695	30	0
88	1	43	602	3969	14518	31943	43819	37638	19720	5888	863	42	0
89	1	44	616	4109	15257	34085	47527	41635	22380	6912	1060	56	0
90	1	44	631	4263	16019	36308	51508	46031	25331	8070	1303	77	0
91	1	45	645	4410	16814	38677	55748	50774	28629	9418	1586	100	1
92	1	45	660	4571	17633	41134	60289	55974	32278	10936	1930	133	1
93	1	46	675	4725	18487	43752	65117	61575	36347	12690	2331	172	2
94	1	46	690	4894	19366	46461	70281	67696	40831	14663	2812	224	3
95	1	47	705	5055	20282	49342	75762	74280	45812	16928	3370	285	5
96	1	47	721	5231	21224	52327	81612	81457	51294	19466	4035	366	7
97	1	48	736	5400	22204	55491	87816	89162	57358	22367	4802	460	11
98	1	48	752	5584	23212	58767	94425	97539	64015	25608	5708	582	15
99	1	49	768	5760	24260	62239	101423	106522	71362	29292	6751	725	22
100	1	49	784	5952	25337	65827	108869	116263	79403	33401	7972	905	30

LEMMA 3. $Q_1(m,N) = Q_1(m,N-m) + Q_1(m-1,N-m)$.

Proof. Lemma 2 can be written in the following way

$$Q_1(m,N) = Q_1(m-1,N-m) + \sum_q Q_1(m-1,N-qm-m).$$

Applying Lemma 2 again for Σ_q we get the asserted result.

Lemmas 2 and 3 show that $Q_1(m,N)$ can be found recursively. The formulas adapt easily to computation using high speed computers. The table on page 92 shows some of the results of such a computation.

It is possible to get closed formulas for $Q_1(3,N)$ and $Q_1(4,N)$ by using the recursive formulas derived. We include the formula for $Q_1(3,N)$ together with a proof for it and that for $Q_1(4,N)$ without its proof.

For N even and $\geqq 6$

$$(1)\qquad Q_1(3,N) = \tfrac{1}{2}\left\{\left[\frac{N}{6}\right]\left(N-1-3\left[\frac{N}{6}\right]\right) + \left[\frac{N-4}{6}\right]\left(N-5-3\left[\frac{N-4}{6}\right]\right)\right\}$$

and for N odd and > 6

$$(2)\qquad Q_1(3,N) = \tfrac{1}{2}\left\{\left[\frac{N-1}{6}\right]\left(N-2-3\left[\frac{N-1}{6}\right]\right) + \left[\frac{N-3}{6}\right]\left(N-4-3\left[\frac{N-3}{6}\right]\right)\right\}$$

where $[x]$ is the integral part of x.

Proof. Lemma 2 for $m = 3$ gives

$$(3)\qquad Q_1(3,N) = \sum_q Q_1(2,N-3q).$$

Evidently

$$(4)\qquad Q_1(2,N) = \frac{N-1}{2} \quad \text{for } N \text{ odd} \geqq 3$$

and

$$(5)\qquad Q_1(2,N) = \frac{N-2}{2} \quad \text{for } N \text{ even} \geqq 4.$$

Introducing new summation indices $q = 2j-1$ for q odd and $q = 2k$ for q even we get from (3)

$$(6)\qquad Q_1(3,N) = \sum_j Q_1(2,N-6j+3) + \sum_k Q_1(2,N-6k).$$

Consider N even. If N is even then $N-6j+3$ is odd and $N-6$ is even. Thus from (4), (5) and (6) we get

$$(7)\qquad Q_1(3,N) = \sum_j \frac{N-6j+3-1}{2} + \sum_k \frac{N-6k-2}{2}$$

for $N - 6j + 3 \geqq 3$ and $N - 6k \geqq 4$ (from (4) and (5)). The lower summation limits are both 1. As j and k are integers only, the upper summation limits are $[N/6]$ and $[(N-4)/6]$. Using the formula for the sum of arithmetic progression in (7) we get the asserted formula (1). By the same argument we get (2).

The closed formula for $Q_1(4, N)$ can be written as follows: Let $(N+n)/4$ be an integer where $n \in \{3, 4, 5, 6\}$. Let further, $a = A(A+1)$, $b = B(B+1)$, $c = C(C+1)$, $d = D(D+1)$, and

$$A = \left[\frac{N-(16-n)}{12}\right] + 1 \qquad B = \left[\frac{N-(20-n)}{12}\right] + 1$$

$$C = \left[\frac{N-(24-n)}{12}\right] + 1 \qquad D = \left[\frac{N-(28-n)}{12}\right] + 1.$$

Then for $n = 6$ and $N \geqq 26$:

$$Q_1(4, N) = \tfrac{1}{2}\{a(4A-5) + 4A + 2b(4B-3) + 4B + 2c(4C+1) + d(4D+3)\};$$

for $n = 5$ and $N \geqq 27$:

$$Q_1(4, N) = \tfrac{1}{2}\{a(4A-5) + 4A + 2b(4B-2) + 2B + 4c(2C+1) + d(4D+5)\}; \tag{8}$$

for $n = 4$ and $N \geqq 28$:

$$Q_1(4, N) = \tfrac{1}{2}\{a(4A-3) + 2A + 2b(4B-1) + 2B + 2c(4C+3) + d(4D+5)\};$$

for $n = 3$ and $N \geqq 25$:

$$Q_1(4, N) = 4a(A-1) + 3A + 4bB + 4c(C+1).$$

The author thanks his colleague Professor N. Vora for valuable discussions of this problem. He also thanks Mrs. B. Thomas for checking the table.

ON PANDIAGONAL MAGIC SQUARES OF ORDER $6t \pm 1$.

CAROLYN BRAUER HUDSON, Duke University, School of Forestry

A magic square of order n is an $n \times n$ square matrix which contains the elements $1, 2, \cdots, n^2$ arranged so that the sum of the elements of each row, each column, and each diagonal is $S = n(n^2+1)/2$. A pandiagonal magic square has the additional property that the sum of the elements of each parallel to the diagonals also equals S. A symmetrical magic square has the property that the sum of $(n-1)/2$ pairs of centrally opposite elements plus the central element also equals S.

A simple method for constructing magic squares of odd order is that of De Lahire (see, for example, Uspensky and Heaslet [1, pp. 160–161]). In this paper we will extend this method to the construction of pandiagonal magic squares and symmetrical pandiagonal magic squares of order $n = 6t \pm 1$, $t = 1, 2, \cdots$.

We denote the elements of the matrix M by m_{ij}. A parallel to the main diagonal for $N - 6j + 3 \geqq 3$ and $N - 6k \geqq 4$ (from (4) and (5)). The lower summation limits are both 1. As j and k are integers only, the upper summation limits are $[N/6]$ and $[(N-4)/6]$. Using the formula for the sum of arithmetic progression in (7) we get the asserted formula (1). By the same argument we get (2).

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Then for $n = 6$ and $N \geqq 26$:

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for $n = 4$ and $N \geqq 28$:

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We denote the elements of the matrix M by m_{ij}. A parallel to the main diagonal is formed by the elements for which $i-j$ is congruent to a given number $h \pmod n$ where $h = 0, 1, \cdots, n-1$, and a parallel to the secondary diagonal is formed by the elements for which $i+j$ is congruent to a given number $h \pmod n$.

Let $a_1, a_2, \cdots, a_n$ be a permutation of the numbers $0, 1, \cdots, n-1$ and let P be the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n-1 & n & 1 & \cdots & n-2 \end{pmatrix}. \tag{1}$$

We set $a_i = a_j$ for $i \equiv j \pmod n$. We then construct the square matrix A by letting the first row be $a_1, a_2, \cdots, a_n$ and applying the permutations $P, P^2, \cdots, P^{n-1}$ successively to the indices of $a_1, a_2, \cdots, a_n$ to get the succeeding rows. Hence

$$A = (a_{ij}) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-1} & a_n & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_3 & a_4 & & a_2 \end{bmatrix}.$$

The indices $\pmod n$ of each row form a complete system of residues $\pmod n$. The same holds for each column since the elements of the ith column are $a_i, a_{i-2}, \cdots, a_{i-2(n-1)}$, and n is odd. The elements of the parallels to the diagonals passing through the element a_i of the first row are $a_i, a_{i-1}, \cdots, a_{i-n+1}$ and $a_i, a_{i-3}, \cdots, a_{i-3(n-1)}$. These indices also form a complete system of residues since $n \not\equiv 0 \pmod 3$. Hence each row, each column, and each parallel to the diagonals consists of the elements $a_1, a_2, \cdots, a_n$ in a certain order.

Now we let $b_1, b_2, \cdots, b_n$ be a permutation of the numbers $1, 2, \cdots, n$, and we construct the matrix $B = (b_{ij})$ by letting the first row be $b_1, b_2, \cdots, b_n$ and applying the permutation P^{-1} to the indices of a row to get the next row. We set $b_i = b_j$ for $i \equiv j \pmod n$. We again have the case that each row, each column, and each parallel to the diagonals is a permutation of the elements $b_1, b_2, \cdots, b_n$.

Now we set

$$M = (m_{ij}) = nA + B. \tag{2}$$

Since the elements of A are the numbers $0, 1, \cdots, n-1$ and those of B are the numbers $1, 2, \cdots, n$, it follows from (2) that $1 \leqq m_{ij} \leqq n^2$. The sum of the elements of each row, column, and each parallel to the diagonals has the value $n^2(n-1)/2 + n(n+1)/2 = n(n^2+1)/2 = S$.

In order to prove that M is the matrix of a pandiagonal magic square we must only show that the elements of M are all different. Now $a_{1j} = a_j$, hence by (1) $a_{2j} = a_{j-2}, \cdots, a_{ij} = a_{j-2(i-1)}$.

Similarly $b_{ij} = b_{j+(2i-1)}$, hence by (2)

$$m_{ij} = na_{j-2(i-1)} + b_{j+2(i-1)}. \tag{3}$$

Now assume that $m_{ij} = m_{rs}$. Then by (3)

$$(4)\qquad na_{j-2(i-1)} + b_{j+2(i-1)} = na_{s-2(r-1)} + b_{s+2(r-1)},$$

$$b_{j+2(i-1)} \equiv b_{s+2(r-1)} \pmod n.$$

Since the b's form a complete system of residues (mod n), it follows that

$$(5)\qquad b_{j+2(i-1)} = b_{s+2(r-1)},$$

$$(6)\qquad j+2(i-1) \equiv s+2(r-1) \pmod n.$$

Substituting (5) into (4) we get

$$a_{j-2(i-1)} = a_{s-2(r-1)},$$

$$(7)\qquad j-2(i-1) \equiv s-2(r-1) \pmod n.$$

Adding and subtracting (6) and (7) it follows that $i = r$ and $j = s$. Therefore each m_{ij} is distinct.

Now we consider a special case of this method to construct a symmetrical pandiagonal magic square. We set $a_1, a_2, \cdots, a_n$ equal to $n-1, 0, 1, \cdots, n-2$ and we set $b_1, b_2, \cdots, b_n$ equal to $2, 3, \cdots, n-1, n, 1$. We construct M in the same way as shown above. If M is symmetrical, then the sum of a centrally opposite pair is n^2+1 and and the central element must be $(n^2+1)/2$. In this case, the central element of A is $(n-1)/2$ and that of B is $(n+1)/2$. Hence from (2) the central element of M is $(n^2+1)/2$.

Now we must show that the sum of each centrally opposite pair is n^2+1. In order to do this we draw x and y axes through the central element of A and label it $(0,0)$ where y is parallel to the columns and x is parallel to the rows, and the positive directions are upward and to the right. Starting at $(0,0)$ and going to $(p,0)$, the x value increases its value by p (mod n), and similarly it decreases by p when going from $(0,0)$ to $(-p,0)$. In the same way the value of y increases by $2q$ (mod n) when going from $(0,0)$ to $(0,q)$ and decreases by $2q$ (mod n) when going from $(0,0)$ to $(0,-q)$. Hence the sum of each pair of centrally opposite elements (p,q) and $(-p,-q)$ in A is

$$(n-1)/2 + p + 2q + (n-1)/2 - p - 2q = n-1,$$

similarly in B

$$(n+1)/2 + p - 2q + (n+1)/2 - p + 2q = n+1,$$

hence n^2+1 in M. It follows that M is a symmetrical pandiagonal magic square.

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The author is currently with the Office of Research and Evaluation, North Carolina Central University, Durham, N. C.

Reference

1. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, New York, 1939.

THE SOLUTION OF A CERTAIN QUARTIC EQUATION

B. FISHER, University of Leicester, England

The quartic equation we are going to solve is the one obtained in the solution of the shed and ladder problem. The problem is to find how high a ladder of length a can reach up a wall with a shed of square cross-section s against it.

If we let this height be h and let the angle the ladder then makes with the horizontal be A we have by trigonometry

$$s(\operatorname{cosec} A + \sec A) = a$$

and then

$$s\left\{\frac{a}{h} + \frac{a}{(a^2 - h^2)^{\frac{1}{2}}}\right\} = a,$$

from which it follows that

$$sh = (h-s)(a^2-h^2)^{\frac{1}{2}}. \tag{1}$$

Squaring (1) we get the quartic equation

$$h^4 - 2sh^3 + (2s^2-a^2)h^2 + 2a^2sh - a^2s^2 = 0. \tag{2}$$

Two of the roots of (2) will be solutions of (1) and the other two roots will be solutions of

$$sh = -(h-s)(a^2-h^2)^{\frac{1}{2}}.$$

Now if h is the solution of the shed and ladder problem then $h' = (a^2-h^2)^{\frac{1}{2}}$ will be the distance of the foot of the ladder from the wall and this is a root of (1). It follows that h and h' are the two roots of (1).

For the case $a < 2^{3/2}s$ the ladder will not reach the wall and so the two roots of (1) will be complex. Considering this case we then have that the two roots of (1) will be complex conjugates of each other. Suppose that

$$h = x + iy, \quad h' = x - iy \quad (x, y \text{ real}).$$

We then have $a^2 = h'^2 + h^2 = z(x^2 - y^2)$. Hence

$$2y^2 = 2x^2 - a^2. \tag{3}$$

Substituting the root $h = x + iy$ into (1) we have

$$s(x + iy) = (x + iy - s)(x - iy).$$

Expansion gives the quadratic

$$4x^2 - 4sx - a^2 = 0$$

with solution

$$x = \tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}}\}.$$

Substituting the positive root for x (otherwise y is imaginary) in (3) we have

$$4y^2 = 2s^2 + 2s(s^2 + a^2)^{\frac{1}{2}} - a^2.$$

It follows that the two roots of (1) are

$$\begin{aligned} x + iy &= \tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}} \pm i[2s^2 + 2s(s^2 + a^2)^{\frac{1}{2}} - a^2]^{\frac{1}{2}}\} \\ &= \tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}} \pm [a^2 - 2s(s^2 + a^2)^{\frac{1}{2}} - 2s^2]^{\frac{1}{2}}\} \end{aligned}$$

and are of course roots of (1) whether (1) has real or complex roots. Thus

$$\tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}} \pm [a^2 - 2s(s^2 + a^2)^{\frac{1}{2}} - 2s^2]^{\frac{1}{2}}\}$$

are two of the roots of (2), the larger of the two roots, for the case $a > 2^{3/2}s$, being the solution of the shed and ladder problem. The other two roots of (2) are given by taking the negative square root of $(s^2 + a^2)^{\frac{1}{2}}$. We thus have that the four roots of (2) are

$$\tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}} \pm [a^2 - 2s(s^2 + a^2)^{\frac{1}{2}} - 2s^2]^{\frac{1}{2}}\},$$

$$\tfrac{1}{2}\{s - (s^2 + a^2)^{\frac{1}{2}} \pm [a^2 + 2s(s^2 + a^2)^{\frac{1}{2}} - 2s^2]^{\frac{1}{2}}\}.$$

THE MATHEMATICAL PROGRAMMING SOCIETY

The Organizing Committee for the Mathematical Programming Society was formed during the Seventh Mathematical Programming Symposium, held at the Hague on September 14–18, 1970. The members of this Committee are: A. Orden (Chairman), J. Abadie, M. L. Balinski (as Editor-in-Chief of the Journal), E. M. L. Beale, A. Charnes, G. B. Dantzig, R. E. Gomory, A. S. Manne, W. Orchard-Hays, M. J. D. Powell, A. Prekopa, A. W. Tucker, S. Vajda, P. Wolfe, and G. Zoutendijk.

The purpose of the Society is the advancement of mathematics, algorithms, applications, and methods of computation in the field which has come to be known as mathematical programming. The scope of this field has been identified in the course of the seven major symposia which have occurred at intervals of about three years since 1951.

In addition to the development of mathematics and algorithms for linear, nonlinear, and integer programming, it is concerned with computer programming and experimentation on algorithms, with applicational models which involve new interpretations of the mathematical structures and processes, and with allied mathematical topics such as unconstrained optimization and graph theory.

The Society will continue the international symposia—heretofore held under

Substituting the positive root for x (otherwise y is imaginary) in (3) we have

$$4y^2 = 2s^2 + 2s(s^2 + a^2)^{\frac{1}{2}} - a^2.$$

It follows that the two roots of (1) are

$$x + iy = \tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}} \pm i[2s^2 + 2s(s^2 + a^2)^{\frac{1}{2}} - a^2]^{\frac{1}{2}}\}$$
$$= \tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}} \pm [a^2 - 2s(s^2 + a^2)^{\frac{1}{2}} - 2s^2]^{\frac{1}{2}}\}$$

and are of course roots of (1) whether (1) has real or complex roots. Thus

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$$\tfrac{1}{2}\{s + (s^2 + a^2)^{\frac{1}{2}} \pm [a^2 - 2s(s^2 + a^2)^{\frac{1}{2}} - 2s^2]^{\frac{1}{2}}\},$$
$$\tfrac{1}{2}\{s - (s^2 + a^2)^{\frac{1}{2}} \pm [a^2 + 2s(s^2 + a^2)^{\frac{1}{2}} - 2s^2]^{\frac{1}{2}}\}.$$

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c/o the International Statistical Institute
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the Hague -- Netherlands

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital C in the margin indicates a classroom review.

Elements of Linear Algebra. By Anthony J. Pettofrezzo. Prentice-Hall, Englewood Cliffs, New Jersey, 1970. 336 pp. $7.95.

This is an orthodox book with no vices, covering vectors in two and three dimensions and elementary matrix theory. At first blush the preceding sentence would appear to damn the book with faint praise. In fact, the large number of unsatisfactory texts indicates that to write an elementary linear algebra text without vices is no mean feat. At any rate, this book is clearly written and easy to understand, contains enough examples and exercises (at the right level) to illustrate the text, and has remarkably few misprints and (unless I am mistaken) no incorrect results. For large first-year classes it has few rivals.

After mentioning "vector quantities", "geometric vectors", "line vectors" and "bound vectors" the author defines "free vector" (essentially as an equivalence class of directed line-segments) and from then on, confines his attention to these. After the basic properties of addition and multiplication-by-scalars, he introduces Cartesian coordinates and the i, j, k notation and throughout, there is a pleasing amount of geometrical applications.

After 110 painless pages the book changes to matrix theory, including determinants and the row-echelon method of solving equations, and the two streams are brought together in chapters on transformations of the plane and eigenvectors.

Altogether a reliable, if unexciting text to put in a student's hands.

HUGH THURSTON, University of British Columbia *ad hoc* sponsorship of diverse organizations—, and in cooperation with the North-Holland Publishing Company, will publish the new journal, MATHEMATICAL PROGRAMMING. Individual subscriptions to the Journal are to be handled by the Society, and institutional subscriptions by North-Holland.

Further information can be obtained from the provisional Secretariat of the Society,

c/o the International Statistical Institute
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HUGH THURSTON, University of British Columbia

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

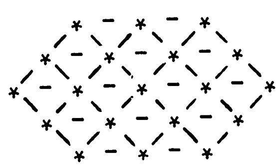
Send all communications for this department to Robert E. Horton, Los Angeles Valley College , 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before September 15, 1972.

PROBLEMS

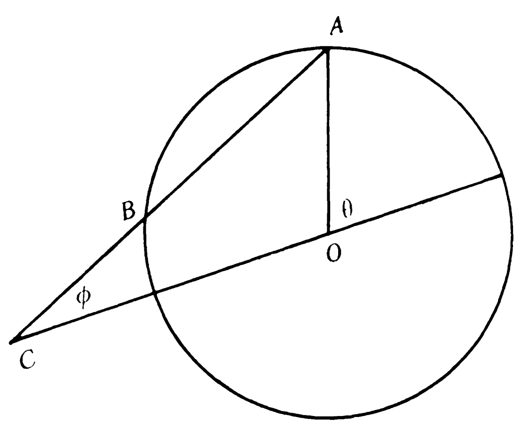
824. *Proposed by Paul S. Lemke, Rensselaer Polytechnic Institute.*

Place the integers 1 through 19 so as to form a "magic hexagon": the sums in each of the fifteen ways indicated are all the same:



825. *Proposed by Henry W. Gould, West Virginia University.*

One method of trisecting an angle uses compasses (to describe a circle with



radius R) and a straight edge with a distance between two marks equal to R. In the figure $CB = R$ with the result that $\phi = \theta/3$.

An incorrect variant of this method uses a straight edge with arbitrary markings such that $CB = BA = x$. In this case establish the relationship between ϕ and θ and determine whether trisection is ever achieved. Extend the discussion to the case $CB = m(BA)$.

826. *Proposed by F. D. Parker, St. Lawrence University.*

Two men, A and B, purchase stock in the same company at times $t_1, t_2, t_3, \cdots t_n$, when the price per share is respectively p_1, p_2, p_3, $\cdots, p_n$. Their methods of investment are different, however: A purchases x shares each time, whereas B invests P dollars each time (we assume it is possible to purchase fractional shares). Show that unless $p_1 = p_2 = \cdots = p_n$, the average cost per share for B is less than the average cost per share for A.

827. *Proposed by V. F. Ivanoff, San Carlos, California.*

Prove that in a triangle with sides a, b, c and angles α, β, γ:

$$\frac{\cot\alpha}{b^2 + c^2 - a^2} = \frac{\cot\beta}{c^2 + a^2 - b^2} = \frac{\cot\gamma}{a^2 + b^2 - c^2}$$

and find the geometric interpretation of the ratios.

828. *Proposed by Warren Page, New York City Community College.*

Call an n-digit number $x = x_1 x_2 \cdots x_n$ an n-linked m-chain if $x_1 + x_n = x_2 + x_{n-1} = x_3 + x_{n-2} \cdots = m$, with $x_{(m+1)/2} = m$ when n is odd. The number 25614 for example is a 5-linked 6-chain.

What is the largest natural number n such that for every n-digit number $x_1 x_2 \cdots x_n$, $x_1 \neq x_n$, $\left| x_1 x_2 \cdots x_n - x_n x_{n-1} \cdots x_1 \right|$ is a k-linked 9-chain, $k \leqq n$?

Can these concepts be extended further?

829. *Proposed by John D. Baum, Oberlin College.*

It is well known that a positive integer can be written as the sum of consecutive integers if and only if it is not a power of two. If a positive integer is so expressible, its representation is not necessarily unique. For example,

$$15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5.$$

For integers of what form are their expressions as sums of positive consecutive integers unique?

830. *Proposed by Frank Dapkus, Seton Hall University.*

Find a right triangle with the smallest area that can be partitioned into two triangles with all integral sides.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q537. Determine solutions to

$$xF'(x) - F(x) = F'(F'(x))$$

other than $F(x) = a(x-1)$.

[*Submitted by Murray S. Klamkin*]

Q538. Show that $\log_e 2$ is irrational.

[*Submitted by E. M. Clarke*]

Q539. If a, m and n are positive integers and n is odd, prove that the greatest common divisor of $a^n - 1$ and $a^m + 1$ is not greater than 2.

[*Submitted by Erwin Just*)]

Q540. If $(x + 1/x)^2 = 3$, evaluate $x^3 + 1/x^3$.

[*Submitted by Miltiades S. Demos*]

Q541. Let X be a connected topological space and x a point in X such that $X - x = A \cup B$ where A and B are separate sets. Give an example where A is not connected.

[*Submitted by Albert White*]

(Answers on page 112)

SOLUTIONS

Late Solutions

R. L. Breisch, Pennsylvania State University: **789**; *V. S. Blanco, University of South Alabama:* **788**; *Derrill J. Bordelon, Naval Underwater Systems Center, Rhode Island:* **791**; *Jaime Machado Cardoso, Universidade Estadual de Campinas, Brazil:* **791**; *David Gilliam, Idaho State University:* **791**; *Heiko Harborth, Braunschweig, Germany:* **788, 789**; *Wells Johnson, Bowdoin College:* **795**; *Patricia M. Shannon, Mutual Savings Life Insurance Company, Decatur, Albama;* **791**; *Jim Tattersall, Attleboro, Massachusetts:* **791.**

The Olympic Runner

796. [May, 1971] *Proposed by Charles W. Trigg, San Diego, California.*

M. Adman Amdam hoped to make the Olympic team in one of the distance events, but he needed a lot of preparatory training,

$$SO/HE = .RANRANRAN\cdots.$$

Each letter in the cryptarithm uniquely represents a positive digit in the scale of nine. Find the only solution less than one-half, and hence more likely to represent his chances of making the team.

I. *Solution* (*in base* 10) *by Merrill Barnebey, Wisconsin State University at LaCrosse.*

In seeking a solution to the problem in positive digits, no two alike, we recall a well-known number theorem to the effect that having factored out powers of two and five from a denominator, the remaining number, c, will determine the period, n, of a repeating decimal by the expression $10^n \equiv 1 \pmod{c}$. Since $n = 3$ here we see that $10^3 \equiv 1 \pmod{999}$ where we also see that $999 = 3^3 \cdot 37$.

Thus we try various numerators divided by 37, less than one-half, but none of two digits gives the desired result. So we try a multiple of 37 (one that results in only two digits) and find $26/74 = .351351351\cdots$.

II. *Solution* (*in base* 9) *by Kenneth M. Wilke, Topeka, Kansas.*

Let $F = \dfrac{SO}{HE} = .\overline{RAN}$. Then $1000F = RAN\,.\,\overline{RAN}$ so that $888F = RAN$. Now $888 = 14 \cdot 62 = 15 \cdot 57 = 28 \cdot 31$ are the only two digit divisors of 888 and hence the only choices for HE. We denote the associate factor of HE as HE'; e.g., $28' = 31$. Now we note that $RAN = SO \cdot HE'$. We exclude 14 and 15 as choices for HE since no two digit choices for SO exist which satisfy $2\,SO < HE$.

$HE \neq 31$ since this implies $S = E$.

$HE \neq 28$ since $SO = 13$ implies $A = S$.

$HE \neq 62$ since $12 \leqq SO < 31$ implies $SO = 13, 14, 15, 17, 18, 30$ and each product $SO \cdot HE' = RAN$ leads to repeated digits.

$HE = 57$ implies $12 \leqq SO < 28$ or $SO = 12, 13, 14, 16, 18, 20, 21, 23, 24, 26$.

Of these, only $SO = 13$ and $SO = 26$ lead to $RAN = 206$ and $RAN = 413$ where no repeated digits occur. Since all digits are positive, $SO = 26$, $HE = 57$, and $RAN = 413$. All calculations are base 9.

Editor's note: Solutions were about evenly divided between base 10 and base 9. Some solvers apparently removed the restriction that the letters must represent positive digits and obtained the solution

$$13/57 = .206206206.$$

Also solved by R. L. Breisch, Pennsylvania State University; Morton Goldberg, Broome Technical Community College, New York; M. G. Greening, University of New South Wales, Australia; Louise S. Grinstein, New York, New York; Brian W. Hogan, Highline Community College, Midway, Washington; John M. Howell, Littlerock, California; J. A. H. Hunter, Toronto, Canada; Dave Logothetti, University of Santa Clara, Calfornia; George B. Miller, Connecticut State College, New Britain, Connecticut; Thomas E. Moore, Bridgewater State College, Maine; C. C. Oursler, Southern Illinois University at Edwardsville; Albert J. Patsche, U. S. Army Weapons Command, Rock Island, Illinois; Fred Pence, Harrisonburg, Virgina; Steven Record, Bucknell University (two solutions); E. P. Starke, Plainfield

New Jersey; Hans Subak, Bentleigh, Victoria, Australia; Paul Sugarman, Massachusetts Institute of Technology; Mark Winholtz, Luther College, Iowa; Francisco Wong, Flushing, New York and the proposer. One incorrect solution was received.

Critical Points

797. [May, 1971] *Proposed by Frank J. Papp, University of Lethbridge, Alberta, Canada.*

Determine the critical points and relative extrema, if any, of the two functions $f(x_1,\cdots,x_n)$ and $g(x_1,\cdots,x_n)$ for $n = 1,2,3\cdots$ where

$$f(x_1,\cdots,x_n) = \det(a_{ij})\, g(x_1,\cdots,x_n) = \det(b_{ij})$$

with

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ or if } i = j = 1 \\ 1 + x_{i-1} & \text{if } i = j = 2,3\cdots n+1 \end{cases}$$

.and

$$b_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 1 + x_i & \text{if } i = j = 1,2,\cdots n\,. \end{cases}$$

Solution by Derrill J. Bordelon, Naval Underwater Systems Center, Rhode Island.

From elementary determinantal operations,

$$\det(a_{ij}) = \prod_{i=0}^{n} x_i,\ \det(b_{ij}) = \prod_{i=0}^{n} x_i \sum_{i=0}^{n} x_i$$

with $x_0 = 1$. Further, $f(x_1,\cdots,x_n) = \det(b_{ij})$,

$$\frac{\partial f}{\partial x_j} = \frac{\prod_{i=0}^{n} x_i}{x_j}\left[\sum_{i=0}^{n}\frac{1}{x_i} - \frac{1}{x_j}\right]$$

and

$$\partial f/\partial x_j = 0 \text{ for all } j = 1,2,\cdots n \Leftrightarrow \sum_{i=0}^{n}\frac{1}{x_i} = \frac{1}{x_j},$$

$j = 1,2,\cdots n \Leftrightarrow x_i = (1-n)$, $i = 1,2,\cdots n$. Therefore, uniquely $x_j = (1-n)$, $j = 1,2,\cdots n$, is a critical point of f.

Since $\partial^2 f/\partial x_j^2 \equiv 0$ then clearly, $(\partial^2 f/\partial x_i x_j)$, being the Hessian of f, is neither negative nor positive definite. Accordingly, $x_j = (1-n)$, $j = 1,2,\cdots n$ is not a relative extremum.

Further, $g(x_1,\cdots x_n) = \det(b_{ij})/\det(a_{ij}) = \sum_{i=0}^{n} 1/x_i\, \partial g/\partial x_j = -1/x_j^2, j = 1,2,\cdots n$ and $\partial g/\partial x_j = 0$ for all $j = 1,2\cdots n$. $\Leftrightarrow -1/x_j = 0$ for all $j = 1,2,\cdots n \Leftrightarrow x_j = \pm\infty$ for all $j = 1,2,\cdots n$. Thus, $x_j = \pm\infty$ for all $j = 1,2\cdots n$ is a critical point. Since $\partial^2 g/\partial x_j^2 = 2/x_j^3, (\partial^2 g/\partial x_j^2)x_j = \pm\infty = 0$ for $j = 1,2,\cdots n$ then $(\partial^2 g/\partial x_i x_j)$, being the Hessian of g, is neither negative nor positive definite. Accordingly, $x = \pm\infty, j = 1,2,\cdots n$ is not a relative extremum of g.

Also solved by E. F. Schmeichel, Itasca, Illinois; and the proposer.

Products of Primes

798. [May, 1971] *Proposed by Peter A. Lindstrom, Genesee Community College, New York.*

Show that $\lim_{x\to\infty}(\prod_{p\leqq x} p)^{1/x} = e$ where $\prod_{p\leqq x} p$ is the product of prime integers that are less than or equal to x.

Solution by Zbigniew Fiedorowicz, Illinois Institute of Technology.

In their proof of the prime number theorem, Hardy and Wright show that

$$\prod(x) \sim \frac{\sum p \leqq x^{\log p}}{\log x}$$

where $\Pi(x)$ is the number of primes less than or equal to x (see Hardy and Wright, *An Introduction to the Theory of Numbers*, p. 345, Theorem 420). By the prime number theorem we have

$$\prod(x) \sim \frac{x}{\log x}.$$

Hence

$$\frac{\sum p \leqq x^{\log p}}{\log x} \sim \frac{x}{\log x}$$

so that

$$\lim_{x\to\infty} \frac{\sum p \leqq x^{\log p}}{x} = 1$$

and

$$\lim_{x\to\infty} (\textstyle\prod_{p\leqq x} p)^{1/x} = \exp\left\{\lim_{x\to\infty} \frac{\sum p \leqq x^{\log p}}{x}\right\} = e.$$

Also solved by Fred Dodd, University of South Alabama; M. G. Greening, University of New South Wales, Australia; Heiko Harborth, Braunschweig, Germany; Vaclav Konecny, Jarvis Christian College, Texas; David E. Manes, Pennsylvania State University; Frank V. Meyer, University of Minnesota; Gary Mullin, Pennsylvania State University; Bob Prielipp and Norbert J. Kuenzi, Wisconsin State University, Oshkosh (jointly); Paul Shimp, Idaho State University; E. F. Schmeichel, Itasca, Illinois; Edward Swem, Cedarbrae Collegiate Institute, Scarborough, Ontario, Canada; C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India; Edwin H. Voorhees, Jr., University of Tennessee, Chattanooga; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Expected Value

799. [May, 1971] *Proposed by N. J. Kuenzi, Wisconsin State University at Oshkosh.*

A population consists of two distinct types of items, n items of type I and m

items of type II. Items are selected randomly one by one without replacement until the kth type I item has been selected, $1 \leqq k \leqq n$. If x_k is the trial on which the kth type I item is selected, find $p[x_p = x]$ where $x = k, k+1 \cdots k+m$ and find the expected value of x_k

Solution by James C. Hickman, University of Wisconsin, Madison.

The n locations for the Type I items in a linear arrangement of the $n+m$ items may be selected in $\binom{m+n}{n}$ ways. The $k-1$ locations for the $k-1$ Type I items in the first $x-1$ locations $(x = k, k+1, \cdots k+k)$ may be selected in $\binom{x-1}{k-1}$ ways. The $n-k$ locations for the remaining Type I items, after the location numbered k is filled with a Type I item, may be selected from among the final $m+n-k$ locations in $\binom{m+n-x}{n-k}$ ways. Therefore,

$$p(x_k = x) = f(x) = \binom{x-1}{k-1}\binom{m+n-x}{n-k} \Big/ \binom{m+n}{n}, \; x = k,$$

$k+1, \cdots k+m$, and the expected number of draws is

$$E(x_k) = \sum_{x=k}^{k+m} xf(x).$$

Let $y = x - k$ and consider the summation

$$\sum_{x=k}^{k+m} x\binom{x-1}{k-1}\binom{m+n-x}{n-k} = k\sum_{y=0}^{m}\binom{y+k}{y}\binom{m+n-y-k}{m-y}.$$

Note that $\binom{y+k}{y}$ is the coefficient of t^y in the expansion of $(1-t)^{-k-1}$ and $\binom{m+n-y-k}{m-y}$ is the coefficient of the t^{m-y} in the expansion of $(1-t)^{-n+k-1}$. Because $1-t)^{-k-1}(1-t)^{-n+k-1} = (1-t)^{-n-2}$, the summation is equal to k times the coefficient of t^m in the expansion of $(1-t)^{-n-2}$. Therefore, $E(x_p) = k\binom{m+n+1}{m}/\binom{m+n}{n} = k(m+n+1)/(n+1)$.

Also solved by Michael E. Bates, Idaho State University; Fred Dodd, University of South Alabama; M. G. Greening, University of New South Wales, Australia; John M. Howell, Littlerock, California; David E. Manes, Pennsylvania State University; Nigel F. Nettheim, Toronto, Canada; Bertram Price, White Plains, New York; Rina Rubenfeld, New York City Community College; E. F. Schmeichel, Itasca, Illinois; Paul Sugarman, Massachusetts Institute of Technology; John R. Ventura, Jr., Naval Underwater Systems Center, Rhode Island; and the proposer.

Gnomon-Magic Squares

800. [May, 1971] *Proposed by David Singmaster, University of London, England.*

In *A property of third order gnomon-magic squares*, this MAGAZINE, 1970, 70, a 3×3 array is called gnomon-magic if the four 2×2 subarrays obtained by removing a 5-element gnomon all have the same sum. Show that a gnomon-magic 3×3 array has its two diagonal sums equal. Does this extend to higher orders?

Solution by Zalman Usiskin, University of Chicago.

Let the gnomon-magic square be

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$$

In the aforementioned article it is shown that $a_1 - a_3 = c_1 - c_3$. Thus $a_1 + c_3 = a_3 + c_1$. Adding b_2 to each side of the equation shows the diagonal sums to be equal.

The array below at left shows that, if we consider a 4×4 square to be gnomon-magic when its 3×3 subarrays have the same sum, the diagonals need not themselves have the same sum. The array below at right shows that, if we consider a 4×4 square to be gnomon-magic when its 2×2 subarrays have the same sum, the diagonals need not themselves have the same sum. Thus the diagonal-sum constancy property for gnomon-magic squares does not extend to all higher orders.

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad\qquad \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Also solved by E. F. Schmeichel, Itasca, Illinois; James J. Tattersall, Attleboro, Massachusetts; Charles W. Trigg, San Diego California; and the proposer.

A Disproved Inequality

801. [May, 1971] *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let x_i be the distance of an interior point of a triangle $A_1A_2A_3$ from the side opposite A_i, $i = 1,2,3$ and let r be the inradius of the triangle. Prove or disprove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \geqq \frac{3}{r}.$$

Solution by L. Carlitz, Duke University.

We shall show that the inequality

$$(*) \qquad \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \geqq \frac{3}{r}$$

is in general not true. Indeed let Q be the point in the interior of the given triangle whose distances to the sides are y_1, y_2, y_3, where

$$\begin{cases} a_1y_1 + a_2y_2 + a_3y_3 = 2K, \\ a_1y_1^2 = a_2y_2^2 = a_3y_3^2, \end{cases}$$

where a_1, a_2, a_3 denote the sides and K the area of $A_1A_2A_3$. We shall show that

$$(**) \qquad \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \geqq \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}$$

with equality only when the given point coincides with Q.

Put

$$M^2 = a_1y_1^2 = a_2y_2^2 = a_3y_3^2.$$

Then

$$2K = a_1y_1 + a_2y_2 + a_3y_3 = M^2\left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}\right).$$

Also

$$2K = M(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3})$$

so that

$$\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} = \frac{(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3})^2}{2K}.$$

Since $a_1x_1 + a_2x_2 + a_3x_3 = 2K$, it follows that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}\right)$$

$$= \frac{x_2x_3 + x_3x_1 + x_1x_2}{x_1x_2x_3} - \frac{(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3})^2}{2K}$$

$$= \frac{(a_1x_1 + a_2x_2 + a_3x_3)(x_2x_3 + x_3x_1 + x_1x_2) - (\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3})^2x_1x_2x_3}{2x_1x_2x_3K}.$$

The numerator of this fraction is equal to

$$a_1x_1^2(x_2 + x_3) + a_2x_2^2(x_3 + x_1) + a_3x_3^2(x_1 + x_2) - 2(a_2a_3 + a_3a_1 + a_1a_2)x_1x_2x_3$$
$$= x_1(\sqrt{a_2}x_2 - \sqrt{a_3}x_3)^2 + x_2(\sqrt{a_3}x_3 - \sqrt{a_1}x_1)^2 + x_3(\sqrt{a_1}x_1 - \sqrt{a_2}x_2)^2 \geqq 0.$$

Equality holds only if

$$\sqrt{a_1}x_1 = \sqrt{a_2}x_2 = \sqrt{a_3}x_3,$$

that is, only if

$$x_1 = y_1, \; x_2 = y_2, \; x_3 = y_3.$$

Also solved H. J. Ader, Amsterdam, The Netherlands; Leon Bankoff, Los Angeles, California;

Mike Bunce, Idaho State University; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; V. F. Ivanoff, San Carlos, California; Lew Kowarski, Morgan State College, Maryland; and C. S. Venkataraman, Sree Kerala Verma College, Trichur, South India.

Fermat's Theorem, Special Case

802. [May, 1971] *Proposed by Erwin Just, Bronx Community College, New York.*

Let p, q and r be distinct odd primes and k, m and t be positive integers. If $p = 2q^k + 1$, $n = r^t$ and $x = p^m$, prove that neither $x^n + y^n = z^n$ nor $x^n - y^n = z^n$ has solutions in positive integers.

Solution by the proposer.

The following lemmas will be used.

LEMMA 1. *If* $(a,x) = 1$, $a^n \equiv b^n \pmod x$ *and* $a \equiv b \pmod x$ *then* $[n, \phi(x)] \neq 1$, (ϕ *is Euler's function*).

Proof of Lemma 1. If $[n,\phi(x)] = 1$, then there exist integers c and d such that $cn + d\phi(x) = 1$. Since $(a,x) = 1$ and $a^n \equiv b^n \pmod x$, it is evident that $(b,x) = 1$. The following congruences may now be easily justified: $a \equiv a^{cn+d\phi(x)} \equiv (a^n)^c(a^{\phi(x)})^d \equiv (b^n)^c \equiv (b^n)^c(b^{\phi(x)})^d \equiv b^{\,cn+a\phi(x)} \equiv b \pmod x$. The conclusion $a \equiv b \pmod x$, however, is incompatible with the hypothesis. This establishes that $[n,\phi(x)] \neq 1$.

LEMMA 2. *If* x, y *and* z *are positive integers and* $x^n + y^n = z^n$, $n \geqq 2$, *then* $2z > x + y > z$.

Proof of Lemma 2. When $n > 1$, $(x + y)^n > x^n + y^n = z^n$. Thus, $(x + y)^n > z^n$ which implies $x + y > z$. Since $x < z$ and $y < z$, it follows that $x + y < 2z$ which completes the proof of the lemma.

Assume that y and z exist such that $x^n + y^n = z^n$. It may be assumed, without loss of generality, that x, y and z are relatively prime in pairs. Since $x^n + y^n = z^n$, it follows that $x^n + y^n \equiv z \pmod x$ or $y^n \equiv z^n \pmod x$. If $[n,\phi(x)] = 1$ then from Lemma 1 it is obtained that $y \equiv z \pmod x$. Since $z > y$ the latter congruence implies that there exists a positive integer k such that $z - y = kx$ or $z = kx + y$. This, in turn, implies that $z \geqq x + y$ which contradicts Lemma 2. It must be the case, therefore, that $[n,\phi(x)] \neq 1$. This is not consistent with the hypothesis, however which requires that $[n,\phi(x)] = (r^t, 2p^{m-1}q^k) = 1$. Therefore, $x^n + y^n = z^n$ has no solutions.

If y and z exist such that $x^n - y^n = z^n$, then $-y^n \equiv z^n \pmod x$ and since n is odd, $(-y)^n \equiv z^n \pmod x$. As in the previous case, it follows from Lemma 1 that since $[n,\phi(x)] = 1$, $-y \equiv z \pmod x$ which implies that for some integer, $k, y + z = kx$; a contradiction, since Lemma 2 requires that $x < y + z < 2x$. This completes the proof.

Also solved by Rina Rubenfeld, New York City Community College; and Phil Tracy, APO, San Francisco, California.

Comment on Problem 787

787. [January and November, 1971] *Proposed by T. J. Kaczynski, Lombard, Illinois.*

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into n^2 little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

Comment by Thomas Wray, Department of Energy, Mines and Resources, Ottawa, Ontario, Canada.

Instead of solving the problem as given, we solve the following generalization: Given an $m \times n \times \cdots \times p$ integral sided N-dimensional rectangular block, let us divide it into $V - mn \cdots p$ *cells* (unit N-cubes) by hyperplanes parallel to its prime faces (i.e., faces of dimension $N - 1$). If we define a *match* to be a unit $(N - 1)$-cube, it is clear that a match is congruent to a prime face of a cell. We wish to place matches on the prime faces of cells in such a way that:

(i) each match exactly covers a prime face of a cell,

(ii) each cell has exactly two prime faces covered by matches, and

(iii) no matches are placed on the prime faces of the block. Find necessary and sufficient conditions on $N, m, n \cdots p$ for a match placing to be possible.

Solution. A match placing is possible if and only if the block has even N-volume V and consists of more than a single stack of cells (implying in particular that $N \geqq 2$). In terms of the sides $m, n \cdots p$, these conditions are that the list $m, n \cdots p$ has at least two terms, that at least one side is even, and that at least two sides are different from 1. The total number of matches required is V.

Remark 1. It is evident that conditions (i) and (iii) imply that matches cover prime faces lying within the block. Hence we have, as a dual to condition (ii), that

(iv) each match covers prime faces of exactly two cells. Two such cells will be called *tied*.

Remark 2. Since condition (iii) prevents the placing of a match in a hyperplane orthogonal to any side of the block of length 1, the dimension N of the block can be altered by the introduction or suppression of sides of length 1 in the list $m, n \cdots p$ without changing the match placing (except, of course, that the dimension $N - 1$ of the matches changes along with N).

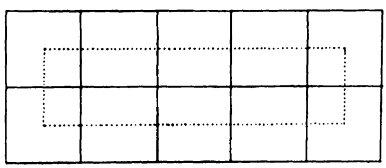
Necessity part of proof. If a match placing exists for a particular block, then conditions (ii) and (iv) imply that each cell has exactly two cells tied to it. If the block consists only of a single stack of cells, an end cell in the stack can have at most one cell tied to it, hence the block has to be effectively (i.e., up to the introduction or suppression of unit sides) at least 2-dimensional for a match placing to be possible.

Now start at any cell; move to either of the cells tied to it. On arriving at a new cell, move to the (unique) cell tied to it, other than the one just left. This defines a path of cells which, since V is finite and since each cell is tied to exactly two others, must close by returning to the original cell. We call such a path a *circuit.* We note that in a circuit there are 1:1 correspondences between cells, matches and steps, each cell corresponding to the match through which it is entered, each step corresponding to the match crossed and to the cell entered.

Two cases can occur: The circuit exhausts all the cells of the block; or it doesn't. In the second case, take another starting cell not on the original circuit and trace out another circuit. Condition (ii) ensures that this new circuit is disjoint from the the original one. Repeating the process a finite number of times (at most $V/4$) if necessary, we decompose the block into a union of disjoint circuits.

Each circuit is a closed path, hence it contains as many steps in each of the $2N$ cardinal directions (parallel to the edges of the block) as it does in the opposite cardinal direction. So the number of steps, hence of matches, and of cells, in the circuit must be even. Since the block is the union of a finite number of disjoint circuits, its total number V of cells must be even. By the 1:1 correspondence the number of matches used is also V.

Sufficiency part of proof. Let $N \geqq 2$, let m be even, let $k = m/2$, and let $n > 1$. Dissect the block into V/kn slabs, each of size $2 \times n \times 1 \times \cdots \times 1$ ($2 \times n$ if $N = 2$). Each slab can be traversed as a circuit as shown below:



The dotted line joins the centers of successive tied cells in the circuit; matches are placed on prime faces cut by the dotted line; the figure can be regarded as depicting a 2×5 slab, as the projection of a $2 \times 5 \times 1 \times \cdots \times 1$ slab on the plane orthogonal to all its unit sides, or, by Remark 2, as $2 \times 5 \times 1 \times \cdots \times 1$ slab with its unit sides suppressed. We obtain the match placing for the whole block by merely reconstructing the block from its constituent slabs, each slab now being equipped with a match placing.

Comments. This generalization is one of the two direct generalizations of the original problem to N-dimensional rectangular blocks with $(N-1)$-dimensional matches. The other direct generalization replaces condition (ii) by

(ii′) each cell has exactly half (i.e., N) of its prime faces covered by matches.

We can generalize further and replace (ii) by

(ii″) each cell has exactly r prime faces covered by matches, for fixed r. Necessary and sufficient conditions for a match placing are now required on $r, N, m, n \cdots p$.

Another generalization consists of lifting the requirement that a match be $(N-1)$-dimensional and that the faces covered be prime. A match is now a unit M-cube, congruent to an M-dimensional face of a cell. The conditions now read:

(i) each match exactly covers an M-face of a cell,

$$\begin{array}{ll} \text{(ii) each cell has exactly} & \\ \text{(ii}'\text{)} & \\ \text{(ii}''\text{)} & \end{array} \left\{ \begin{array}{l} \text{two} \\ \text{half. i.e., } 2^{N-M-1}\binom{N}{M} \text{ of its} \\ r\text{, for fixed } r \end{array} \right\}$$

M-faces covered by matches, and

(iii) no matches are placed on the boundary of the block. What are the necessary and sufficient conditions on r (in case ii''), $M, N, m, n \cdots p$ for a match placing to be possible? A natural specialization of this generalization is the case $M = 1$, where a match is a unit segment and covers an edge of a cell.

ANSWERS

A537. This is a Clairaut equation. Consequently we differentiate obtaining $xF''(x) = F''(x)F''(F'(x))$. One solution is $F''(x) = 0$ or $F(x) = a(x-1)$. The other solutions are derivable from $x = F''(F'(x))$. The general solution of this latter equation seems difficult to derive. However, it does have the power solution $F(x) = ax^{n+1}$ where $n = (1 \pm 5)/2$ and $a = n^{-1/n}/(n+1)$.

A538. Suppose $\log_e 2$ is rational. Clearly $\log_e 2 \neq 0$, hence $\log_e 2 = p/q$ where p and q are integers, $p > 0$ and $q \neq 0$. Thus $e^{p/q} = 2$ or $e^p = 2^q$. This implies that e satisfies the equation $x^p - 2^q = 0$ and this is a contradiction since e is known to be transcendental.

A539. Let $d = (a^n - 1, a^m + 1)$. Then for some integers k and r, $a^n = kd + 1$ and $a^m = rd - 1$. Therefore

$$a^{mn} = (a^n)^m = (kd+1)^m = td + 1$$

for some integer, t, and

$$a^{mn} = (a^m)^n = (rd-1)^n = ud + 1$$

for some integer u.

Thus $td + 1 = ud - 1$ or $(u-t)d = 2$, it follows that $d = 1$ or $d = 2$.

A540.
$$\begin{aligned} x^3 + 1/x^3 &= (x + 1/x)^3 - 3(x + 1/x) \\ &= (x + 1/x)[(x + 1/x)^2 - 3] \\ &= 0. \end{aligned}$$

A541. Let $X = X$, x the cut point in X, A the two upper branches of x, and B the two lower branches of X.

(Quickies on page 102)

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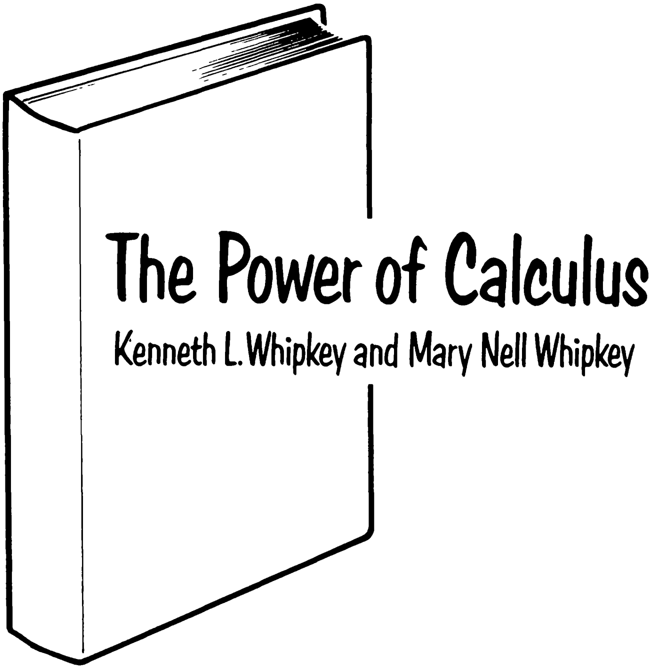

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M. Wiles Keller, Purdue University

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